452 Hydrodyn. (Bethe)

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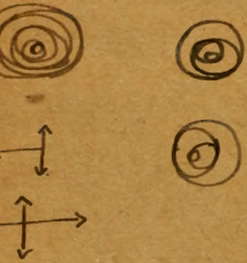
$$M = \sum r_i \times p_i$$

$$\dot{M}_4 = \sum r_i \times \dot{p}_i$$

$$= \sum r_i \times \sum f_{ij}$$



Physics 452; Hydro- & Aero-dynamics. Prof. Bethe.

Books: Lamb: Classical; i.e. up to 1910. No modern developmt
Mathematical approach.

Collection on "Aerodynamic theory" - Prandtl, Taylor on (7 vol) Viscous & Compressible fluids, respectively - entirely from physical viewpoint.

Collection "Modern devel. of fluid dynamics" - (hydro-). (2 vol)

"Fluid mech. for hydraulic engineers" - more elementary

Prandtl & Tietjens -

Report to Nat'l Res. Council - Bateman etc... Survey

"Secret" books: [blue green] Handbuch der Physik Vol VII - good theoretical articl.

" " Expt. Physik. vol 4.

Feb. 2

Classical hydrodynamics: problems treatable by potential theory - i.e. 1) irrotational motion - $\nabla \times \underline{v} = 0$ If motion is everywhere irrotational, including boundaries, can show there should be no resistance to motion thru fluid; even including viscosity terms. However no real fluid is irrotational.

2) Stokes law: resistance ∼ velocity for small velocities - No longer holds for large velocities.

3) Acoustics unlike 1) & 2), assumption of compressibility is made - also, all velocities << sound velocity. neglect 2nd order terms

Modern: Boundary layer theory of Prandtl. Takes care of fluids moving past around bodies — neglect

certain terms near body; others elsewhere — Problem of body of arbitrary shape moving is still unsolved; even sphere isn't completely solved. Prandtl & Karman chief investigators.

Acoustics for large amplitude: shock waves. & velocities ≈ sound

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Classical theory can give useful results; e.g. in case of sphere moving thru fluid, get "potential flow" in front, & behind at sides; but not in wake. — (get vortices & turbulent flow).

Equation of motion found by Euler; in present form, by Stokes. (Stokes included viscosity).

No viscosity: State described by pressure — uniform at each pt. (same in any direction) — $p = \frac{\text{force}}{\text{area}}$ — a scalar f'n of coord. $\underline{r}$ & time t.

density ρ, a function of $(\underline{r}$ & $t)$ also. In all simple cases, simple relation between ρ & p; Particularly simple case: density indep. of pressure (incompressible) e.g. water up to 100 atm. pressure. For gas, ρ definitely not constant. In many applications, can assume adiabatic changes for gases; ($p v^{\gamma} = \text{const}$ for ideal gas) $p \sim \rho^{\gamma}$. $\gamma = \frac{c_p}{c_v} = 1$ to $5/3$.

Also need $\underline{v}$ material velocity, with components u, v, w. $\underline{v}(\underline{r}, t)$.

One relation holds without consideration of forces — i.e. the continuity equation. If ρ is const in volume V, then amount of material is ρV; this changes with time; the change = amount which flows out, i.e. $\frac{\partial}{\partial t}\int \rho \, d\tau = -\int_S \rho v_n \, d\sigma = \int \nabla \cdot \rho \underline{v} \, d\tau = (\nabla \cdot \rho \underline{v}) V$

let $V \to 0$: $\boxed{\dot{\rho} + \nabla(\rho \underline{v}) = 0}$

this expresses conservation of matter.

For incompressible liquid, ρ = const and $\boxed{\nabla \cdot \underline{v} = 0}$

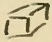
In general: $\boxed{\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho + \rho \nabla \cdot \underline{v} = 0}$

Euler equation of motion:

small volume element $d\tau$

$\rho \, d\tau$ = am't of material. want its acceleration

Newton: $(\rho \, d\tau) \frac{d\underline{v}}{dt}$ = Force

Forces of two kinds: "body" force like gravitation – prop. to mass of body. ∴ convenient to speak of force per unit mass. $\underline{F}$. Also force due to pressure: simple expression for uniform pressure on parallelopiped:  $F_x = (p_x - p_{x+dx})\, dy\, dz$

$= -\frac{\partial p}{\partial x}\, dx\, dy\, dz$

$= -\frac{\partial p}{\partial x}\, d\tau$

$\underline{F} = -\nabla p \, d\tau$

Hence equation becomes:

$$\rho \, d\tau \frac{d\underline{v}}{dt} = \rho \, d\tau \, F - d\tau (\nabla p)$$

Euler eq'n or $\boxed{\frac{d\underline{v}}{dt} = \underline{F} - \frac{1}{\rho} \nabla p}$.

Feb. 4. $\frac{d\underline{v}}{dt}$ is for a given element of matter. not same as was taken in deriving continuity eq'n. Will denote always by $\frac{\partial}{\partial t}$, derivative at a given pt. in space. $\underline{r}$. Will denote by $\frac{d}{dt}$, derivative for a given material element which moves with velocity $\underline{v}$ thru space.

$$\text{Now} \quad \frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial \varphi}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial \varphi}{\partial z} \frac{dz}{dt}.$$

$$= \frac{\partial \varphi}{\partial t} + \underline{v} \cdot \text{grad}\, \varphi$$

Hence Euler eq'ns become: $\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \text{grad}\, \underline{v} = \underline{F} - \frac{1}{\rho} \nabla p$

x-component: $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = F_x - \frac{1}{\rho} \frac{\partial p}{\partial x}$

Hydrostatics

Fluid at rest, $v = 0$

then $F = \frac{1}{\rho}\nabla p$.

In general F is known, p calculated from above eq'n.

A) If also ρ is constant, and F is a conservative force,

i.e. $F = -\nabla V$, then $\nabla(V + p/\rho) = 0$

and $V + p/\rho = \text{const.}$

If $\underline{F} = -g\underline{k}$ (gravity), then $V = gz$

whence $p = -\rho g z + \text{const.}$

(hydrostatic paradox)

$1 \text{ atm.} = 76 \text{ cm Hg} = 1033 \text{ cm water} = 1.012 \cdot 10^6 \text{ dyne/cm}^2$

$= 1.012 \text{ bar}$.

B) Suppose now ρ varies: (e.g. atmosphere). In general, $\rho = \rho(p)$

$$-g = \frac{1}{\rho}\frac{dp}{dz}.$$

then $-gz = \int \frac{dp}{\rho} = \omega - \text{const.}$

$\omega = \text{const} - gz$.

(Atmosphere, (nearly) : $\rho = \frac{\rho_0}{p_0} p$

$\omega = \frac{p_0}{\rho_0}\int \frac{dp}{p} = \frac{p_0}{\rho_0} \ln p/p_0$

Whence $\ln p/p_0 = \text{const.} - \frac{\rho_0}{p_0} g z$

if $p = p_0$ at $z = 0$ (e.g. atm. at sea level)

then $\ln p/p_0 = -\frac{\rho_0}{p_0} g z$

$$\therefore p = p_0 e^{-\frac{\rho_0}{p_0} g z}.$$

f) Liquid rotating as rigid body. (incompressible)

$$\underline{F} = -g\,\underline{k} + \omega^2 \underline{r}_1 \qquad (\underline{r}_1 = \text{comp. of } \underline{r} \perp \text{ to axis of rotation})$$

then $V = gz - \frac{1}{2}\omega^2 r_1^2$

if axis is vertical, $r_1^2 = x^2+y^2$, $\underline{a}_1 \perp \underline{k}$:

$$V = gz - \tfrac{1}{2}\omega^2(x^2+y^2).$$

then $p = \text{const} - \rho V$

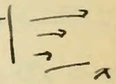
surfaces of constant pressure are paraboloids of revolution

free surface is then also a paraboloid.

Viscosity.

$\underline{v} = 0$ at wall. if wall is at rest; otherwise $\underline{v} = \underline{v}_{wall}$

Hence in general, there is a difference in velocity between various parts of fluid - i.e. a velocity gradient. Hence there is a force - tangential along a boundary $\perp$ to velocity gradient - i.e. a shear stress - expect shear τ_{xy} to be proportional to velocity gradient $\frac{\partial u}{\partial y}$



write: $\tau_{xy} = \eta \frac{\partial u}{\partial y}$.

dimensions: $\left[\frac{dyne}{cm^2}\right] = [\eta]\left[\frac{cm/sec}{cm}\right]$

$\therefore [\eta] = \left[\frac{dyne\text{-}sec}{cm^2}\right] = \left[\frac{gm}{cm\text{-}sec}\right]$

water: $\eta \sim .01$ in this unit

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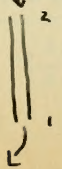
$\tau_{xy} = \eta \frac{\partial v_x}{\partial y}$.

Flow thru cylindrical tube: Poiseuille.



if falling under own weight,

then $p_2 - p_1 = \rho g h$

Consider a cylinder of fluid, of radius y;

What are forces acting on it? : pressure, top + bottom, viscous forces on sides. Net pressure force : $(p_2 - p_1)\pi y^2$.

assume shear constant over length ?infinity; then total shear force is given by : $\tau \cdot 2\pi h y$. hence $(p_2-p_1)\pi y^2 + \tau \, 2\pi h y = 0$

(τ is abs. magnitude)

$\therefore \tau = -\frac{1}{2}(p_2-p_1)\, y/h$.

(Make tube horizontal)

but $\tau = \eta \frac{\partial v}{\partial y}$.

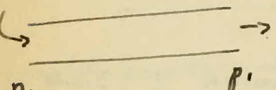
for ?vigor.

$\therefore \frac{\partial v}{\partial y} = -\frac{1}{2}\frac{p_2-p_1}{\eta h}\cdot y$

(v is velocity to right)

τ pos. to right.

$v = -\frac{1}{4}\frac{p_2-p_1}{\eta h} y^2 + \text{const.}$



but $v = 0$ at $y = r$,

$(p_2 > p_1)$

$\therefore v = \frac{p_2-p_1}{4\eta h}(r^2 - y^2)$

then total flow per unit time is :

$$\int_0^r v \cdot 2\pi y \, dy$$

$$= 2\pi \cdot \frac{p_2-p_1}{4\eta h}\int_0^r (r^2-y^2)\, y \, dy \quad \Big\} = \int_0^{r^2} u \frac{du}{2}$$

$$= \frac{\pi(p_2-p_1)}{2\eta h}\cdot\frac{1}{4}r^4$$

$$= \frac{\pi(p_2-p_1)r^4}{8\eta h}.$$

From this can measure η with good accuracy ~1%. liquids - (not so convenient for gases).

water : $\eta \sim .100$ c.g.s. } ..
air : $\eta \sim 10^{-4}$

General velocity distribution: Viscous forces depend on nine quantities $\frac{\partial v_i}{\partial x_j}$. They form a dyadic — transform under rotation of axes like a dyadic. Can find particular combinations:

$(\nabla \cdot \underline{v})$ is invariant under rotations. (scalar).

$(\underline{\nabla} \times \underline{v})$ (curl $\underline{v}$) - transforms like an axial vector under rotation.

Can also form a tensor: (symmetrical)

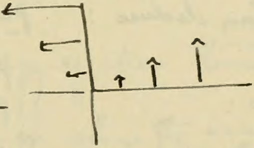
$$\begin{vmatrix} \frac{\partial v_x}{\partial x} - \frac{1}{3}\nabla\cdot v & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\ \text{etc.} & \frac{\partial v_y}{\partial y} - \frac{1}{3}\nabla\cdot v & \\ & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} - \frac{1}{3}\nabla\cdot v \end{vmatrix}$$

(these five transform like $P_{2m} e^{im\phi}$ under rotation).

Physical significance: suppose $v_x = ax$: the fluid becomes extended in x-direction; likewise $\frac{\partial v_y}{\partial y}$ means extension in y-direction, $\frac{\partial v_z}{\partial z}$ in z-direction; to remain incompressible, $\nabla\cdot\underline{v}$ must vanish, i.e. extensions in one direction are compensated by contractions in other directions. Hence sensible to separate this from other comp's.

Suppose $\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \neq 0$ other comp. of tensor $= 0$.

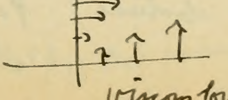
then $\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} = 0$; then $v_y = ax$, $v_x = -ay$.



corresponds to rotation of fluid as rigid body about origin. No viscous forces.

Next: $\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \neq 0$ then $v_y = ax$, $v_x = ay$

$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0$



deforms ∟ to ∠

viscous forces [illegible]

Consider stress tensor.

Force across any boundary in a fluid is not in general normal to boundary: Stress is given by a symmetrical tensor whose spur in general does not vanish. Diagonal components, σ_{xx} σ_{yy} σ_{zz},

Non diagonal: τ_{xy}, τ_{yx}, τ_{zx}.

If boundary has normal $\underline{n}$, then Force $\underline{p_n}$ across plane | $p_{n_x} = \sigma_{xx}\cos(n,x) + \tau_{xy}\cos(n,y) + \tau_{xz}\cos(\ldots$ etc.

Feb. 9, '42 | Assume medium isotropic: this will determine relation between stress tensor & deformation tensor almost completely.

Surface with normal $\underline{n}$; $\underline{p_n}$ = force across surface.

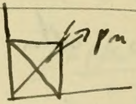
if $\underline{n}$ is in x-direction, call components of $\underline{p_x}$: p_{xx}, p_{xy}, p_{xz}. [or: σ_x τ_{xy} τ_{xz} (elasticity notation)]

similarly $\underline{p_y}$ acting across boundary ⊥ y direction.

comp. p_{yx} p_{yy} p_{yz} etc.

or τ_{yx} σ_y τ_{yz}

Consider equilibrium of:



Can deduce: $\underline{p_n} = \underline{p_x}\cos(n,x) + \underline{p_y}\cos(n,y) + \underline{p_z}\cos(n,z)$.

i.e. $p_{n_x} = p_{xx}$ " $+ p_{yx}$ " $+ p_{zx}$ "

$p_{n_y} = p_{xy}$ " $+ p_{yy}$ " $+$ " "

$p_{n_z} =$ " " " "

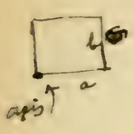
[Can write as:

$\underline{p_n} = \underline{P}\cdot\underline{n}$.]

$\underline{P}$ is stress tensor.

Can also deduce: $p_{xy} = p_{yx}$ etc.

Consider equilibrium against rotation:



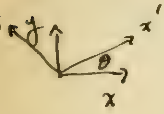
Moment $= \underline{r} \times \underline{F}$ $\qquad M_z = x F_y - y F_x$.

then for equilibrium:

$$a \cdot p_{xy} \cdot bc - b \cdot p_{yx} \cdot ac = 0$$

$$\therefore p_{xy} = p_{yx} .$$

i.e. stress tensor is symmetric.



rotate thru angle θ.

then $\underline{p}_{x'} = \underline{p}_x \cos\theta + \underline{p}_y \sin\theta$

$$p_{x'x} = p_{xx} \cos\theta + p_{yx} \sin\theta$$

component of $\underline{p}_{x'}$ in direction x' is:

$$p_{x'x'} = p_{x'x} \cos\theta + p_{x'y} \sin\theta$$

$$= p_{xx} \cos^2\theta + 2 p_{xy} \cos\theta \sin\theta + p_{yy} \sin^2\theta$$

likewise $p_{x'y'} = -(p_{xx} + p_{yy}) \cos\theta \sin\theta + p_{xy}(\cos^2\theta - \sin^2\theta)$

$$p_{y'y'} = p_{xx} \sin^2\theta - 2 p_{xy} \cos\theta \sin\theta + p_{yy} \cos^2\theta .$$

$$\left\{ \begin{aligned} p_{x'x'} + p_{y'y'} &= p_{xx} + p_{yy} \\ p_{x'x'} - p_{y'y'} &= (p_{xx} - p_{yy}) \cos 2\theta + 2 p_{xy} \sin 2\theta \\ 2 p_{x'y'} &= (p_{xx} - p_{yy}) \sin 2\theta + 2 p_{xy} \cos 2\theta \end{aligned} \right.$$

sum of diag. comp. of stress tensor is unchanged under rotation; other comp. change to each other.

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3 dim: p_{xy}, p_{yz}, p_{zx} $\rightarrow$ transform into each other —

$(p_{xx} - p_{yy}), (2p_{zz} - p_{xx} - p_{yy})$ $\rightarrow$ " " " "

(like P ...) $\quad \frac{1}{3}(p_{xx} + p_{yy} + p_{zz})$ invariant under any rotation.
$= p$

they transform like corresponding bilinear form : $x^2+y^2+z^2$ inv.,
others like xy, yz, zx ; x^2-y^2, $2z^2-x^2-y^2$,

Since stress tensor is symmetric, it can be related to symmetric part only of deformation tensor; (not to antisym (i.e. curl $\underline{v}$) part).

invariant part of stress must be related to invariant part of deformation

in fact, 5-tensor of stress is proportional to 5-tensor of deformation,

[since rotation ——————? ?]

$$\therefore\ p_{xy} = \eta\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \qquad \eta = \eta(\rho, T).$$

etc.

$$p_{xx} - p_{yy} = \eta\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right)$$

etc.

$$p_{xx} - \tfrac{1}{3}(p_{xx}+p_{yy}+p_{zz}) = \eta\left(\frac{\partial v_x}{\partial x} - \tfrac{1}{3}\nabla\cdot\underline{v}\right).$$

$p = p(\rho) + \kappa\, \nabla\cdot\underline{v}$ ordinarily second term is small. $\nabla\cdot\underline{v}$ very small for nearly incomp. fluid.

In general no relation between κ + η.

Oct. 11 '42

$$p_{ik} + p\,\delta_{ik} = \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \tfrac{2}{3}\nabla\cdot\underline{v}\,\delta_{ik}\right)$$

$-p = \frac{1}{3}(\Sigma p_{ii})$ $\qquad p = p_0(\rho, T) + \kappa\, \nabla\cdot\underline{v}$

$$\therefore\ p_{ik} = \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) - \delta_{ik}\left[p_0(\rho, T) + (\kappa + \tfrac{2}{3}\eta)\nabla\cdot\underline{v}\right]$$

e.g. $$\sigma_x = 2\eta\frac{\partial v_x}{\partial x} - \left(p_0 + (\kappa + \tfrac{2}{3}\eta)\nabla\cdot\underline{v}\right)$$

$$\tau_{xy} = \eta\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$

etc.

Want to find eq'n's of motion.

$p_n = \underline{\underline{p}} \cdot \underline{n}$

Consider volume element: Total force due to stress

is $\int \underline{\underline{p}} \cdot \underline{n} \, d\sigma$

By Gauss' theorem $= \int d\tau \, \nabla \underline{\underline{p}}$

(really apply to each comp. separately) $= (\Delta\tau) \, \nabla \cdot \underline{\underline{p}}$.

Force per unit volume $= \nabla \cdot \underline{\underline{p}}$.

[e.g. $F_x = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$.

Hence eq'n of motion becomes:

$$\rho \frac{d\underline{v}}{dt} = \rho \underline{F} + \nabla \cdot \underline{\underline{p}}$$

x.comp: $$\rho \frac{dv_x}{dt} = \rho F_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$$

$$= \rho F_x + \frac{\partial}{\partial x}\left[2\eta \frac{\partial v_x}{\partial x} - (p_0 + (\kappa + \tfrac{2}{3}\eta)\nabla \cdot \underline{v}\right] + \frac{\partial}{\partial y}\left[\eta\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\eta\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\right]$$

$$= \rho F_x + \eta \nabla^2 v_x + \eta \frac{\partial}{\partial x}(\nabla \cdot \underline{v}) - \frac{\partial p_0}{\partial x} + \frac{\partial}{\partial x}(\kappa + \tfrac{2}{3}\eta)\nabla \cdot v$$

$$\rho \frac{dv_x}{dt} = \rho F_x + \eta \nabla^2 v_x - \frac{\partial p_0}{\partial x} - (\kappa + \tfrac{1}{3}\eta)\frac{\partial}{\partial x}\nabla \cdot \underline{v}$$

$$\rho \frac{d\underline{v}}{dt} = \rho \underline{F} + \eta \nabla^2 \underline{v} - \nabla p_0 - (\kappa - \tfrac{1}{3}\eta)\nabla(\nabla \cdot v)$$

$$= \rho \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla)\underline{v}$$

or $$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla)\underline{v} = \underline{F} + \frac{\eta}{\rho}\nabla^2 \underline{v} - \frac{1}{\rho}\nabla p_0 - \frac{1}{\rho}(\kappa + \tfrac{1}{3}\eta)\nabla(\nabla \cdot \underline{v})$$

the last term is negligible usually (∇·v small); drop it. also drop subscript o from p_0. then have Navier–Stokes eq'ns.

the second order term $(\underline{v} \cdot \nabla)\underline{v}$ gives all the trouble.

if $\nabla^2 \underline{v}$ is neglected, the Euler equations are obtained.

$\nabla^2 \underline{v} = \nabla(\nabla\cdot\underline{v}) - \nabla\times(\nabla\times\underline{v})$

but since we have neglected terms in $\nabla\cdot\underline{v}$, we have:

$$\dot{\underline{v}} + (\underline{v}\cdot\nabla)\underline{v} = \underline{F} - \frac{1}{\rho}\nabla p \mp \frac{\eta}{\rho}\nabla\times(\nabla\times\underline{v}) .$$

Irrotational motion: $\nabla\times\underline{v} \equiv 0$; No viscous force.

Then $v = -\nabla\phi$; Classical hydrodynamics;

this is why classical theory fails; because walls always require $v=0$ somewhere; & require also a change in velocity (tangential) in direction normal to wall - i.e. curl $\underline{v} \neq 0$.

Similarity law: if one solution is known, another one can be found if scale is changed throughout.

$\frac{v^2}{l}$	$(v\cdot\nabla)v$ has dimension $\frac{cm^2}{cm\ sec^2} = cm/sec^2$
$\frac{\eta}{\rho}\frac{v}{l^2}$	$\frac{\eta}{\rho}\nabla^2 v$ " " $\frac{gm}{cm\ sec}\cdot\frac{cm^3}{gm}\frac{cm}{sec\cdot cm^2} = cm/sec^2$

the quantity $R = \frac{\rho v l}{\eta}$ is dimensionless; ρ, η, v, are determined, need only pick a characteristic length l of the system. e.g. flow in tube, l could be radius, v " " vel. at center.

R: Reynolds no. e.g. sphere moving in fluid: l could be diameter, v " " velocity of body.

e.g. fluid moving along plate: v could be velocity of fluid at large distance from plate., x distance travelled -

A solution of equation remains a solution if R is the same. (as far as $(v\cdot\nabla)v$ & $\frac{\eta}{\rho}\nabla^2 v$ are concerned) but P/ρ must be changed in same ratio as $(\underline{v}\cdot\nabla)\underline{v}$

Suppose all velocities multiplied by a :
" linear dimensions " " b .

Then if R is to be the same, $\frac{\eta}{\rho}$ must be changed in ratio ab.

$\frac{p}{\rho}$ changes in ratio a^2. (really pressure differences)

Time change in ratio b/a

Hence motion (e.g.) of high viscosity fluid with a real low-viscosity fluid with small dimensions; or hi-velocity motion can be found with low-velocity fluid & different viscosity etc.

Feb. 13 1941

Stokes eq'n. $$\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v} = \underline{F} - \frac{1}{\rho}\nabla p - \frac{\eta}{\rho}\nabla\times\nabla\times\underline{v}$$

Boundary cond: $v = 0$ at walls.

if $\underline{R}$ is kept constant, get the same solution for a new problem with different sizes, velocities, viscosities, etc.

$\underline{R}$ represents ratio of Euler inertia term $\{(\underline{v}\cdot\nabla)\underline{v}\}$ to viscosity term $\left[\frac{\eta}{\rho}\nabla^2\underline{v}\right]$; hence if R is small, viscous term predominates; $(\underline{v}\cdot\nabla)\underline{v}$ negligible; if R is large, inertial term predominates, $\frac{\eta}{\rho}\nabla^2\underline{v}$ [illegible]

if R is small, get 2nd order, 1st degree equation, which is often soluble. (e.g. Stokes's law). valid up to $R \approx 10$.

Stresses $\sim \eta\,\nabla\cdot\underline{v} \sim \eta\frac{v}{l}$. $\therefore$ Force $\sim \eta\frac{v l^2}{l} = \eta v l$.

if R is large, get viscous term still can't be neglected, esp. if we want to know about resistance to motion. Turns out that viscous term may be neglected over most of fluid, but not near body, where velocity changes quite rapidly.

$\frac{\partial v}{\partial t}$ gives pressures ~ acceleration of body;
if v is constant, only pressures are from Euler term $(v\cdot\nabla)v$, which gives pressures ~ v^2; hence reversing of all velocities gives same forces as before; hence must give no force (otherwise it could produce acceleration, which is absurd). ∴ Euler term gives no net ~~eff~~ resistance for uniformly moving body. For large R, classical flow (non viscous) except in a "boundary layer" surrounding solid bodies. For steady flow, neglecting body forces, get pressures $p \sim \rho v^2$, Force ~ $\rho v^2 l^2$; as long as ~~...~~ body is not streamlined) [vortices are formed at back, & classical hydrodyn. doesn't hold.].

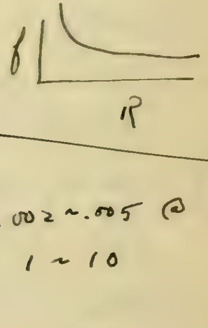
Hence $\frac{\text{Force (large R)}}{\text{Force (small R)}} \sim \rho\frac{v^2 l^2}{\eta v l} = \rho\frac{vl}{\eta} = R$. (actually about R/10)

~~For intermediate~~

In general: Resistance $= \rho v^2 l^2 f(R)$.

large R : $f(R) \sim 1$

small R : $f(R) = 1/R$.



Feb. 16

Water:

$\eta = .01032$ poise @ 20°C. $[\text{poise}] = \left[\frac{gm\ cm}{sec}\right]$.

50°C : $= .00550$

100°C : $.00297$

Hg : 20° : .0156 — organic fluids, many ~ .002 ~ .005 @ 20°

C_2H_5OH 20° : .0121 — oils : ~ 1 ~ 10

liquid CO_2 : .0007 at −30°.

Glacier : ~~...~~ $\sim 10^{10}$.

−100 .87

Gases : air @ 20° : $1.8\cdot10^{-4}$ — η/ρ about 15 times larger for air than water!

100° 2.2

gases : η increases with T; liquids, decreases; mechanism!

is different; gas viscosity: momentum transport across boundary greater at higher temp., since momentum is greater; liquids, tendency of molecules to slip into holes left by others; slip easier at high T, ∴ less viscosity. Liquid viscosity behaves roughly as $Ae^{+Q/T}$. kB is "activation energy".

Reynolds number: water, $\eta/\rho \sim .01$, $\rho/\eta \sim 100$.

$$R = 100 \cdot l \cdot v.$$

if $l \sim 1$ cm, $v \sim 1$ cm/sec, R is 100, which is large

inertial terms more important than viscosity terms.

if $l \sim 1$ meter, $v \sim 10$ m/sec, $R \sim 10^7$ enormous.

To get small R, need, e.g. $l \sim 10^{-3}$ $v \sim 10^{-1}$ (dust particle falling thru liquid) $R \sim 10^{-2}$; Millikan oil drops: $R \sim 10^{-3}$

then can neglect inertial terms.

But for small particles, have to be careful about considering gas as a continuum; if particle size ~ mean free path (10^{-5} cm) can't use hydrodynamical theory.

Return to hydrodyn. eq'ns.

Continuity: $\dot{\rho} + \nabla \cdot \rho \underline{v} = 0$

or: $\dot{\rho} + \underline{v} \cdot \nabla \rho + \rho \nabla \cdot \underline{v} = 0$

or: $\frac{d\rho}{dt} + \rho \nabla \cdot \underline{v} = 0$

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla \cdot \underline{v}.$$

Eq'n of motion: $\dot{\underline{v}} + (\underline{v} \cdot \nabla)\underline{v} = \underline{F} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \underline{v}$

Four eq'ns in 4 unknowns: $\underline{v}$, p.

ρ assumed uniquely related to p: $p = p(\rho)$.

"equation of state".

[In general, $p = p(\rho, T)$, then need another eq'n. for T; comes from conductivity & energy conservation].

Present approximation: assume ρ constant.

$$\text{then } \nabla\cdot\underline{v} = 0 .$$

2nd approx: $\rho(p)$ ρ a f'n of p alone; this is alright until motion becomes violent.

3rd approx: $\rho(p, T)$. (exact).

I ncompressible fluid.

Exact Solutions: i.e. neglect no terms.

Assume fluid moving in x-direction; $v_y = v_z = 0$. $v_x = u$.

Follows that u depends only on y & z : $u(y, z)$. [since $\frac{\partial u}{\partial x} = 0$ from continuity]

$$\text{then } \frac{\partial u}{\partial t} + (\underline{v}\cdot\nabla) u = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\eta}{\rho}\nabla^2 u$$

$$x: \quad \frac{\partial u}{\partial t} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\eta}{\rho}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad \text{since } (v\cdot\nabla) u = 0$$

$$y: \quad 0 = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$z: \quad 0 = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z}.$$

if $\underline{F} = -\nabla V$, then $\frac{\partial V}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$,

$$\therefore V = -\frac{p}{\rho} + f(x)$$

$$\text{or } p = -\rho V + f(x) .$$

$$\text{then } \frac{\partial u}{\partial t} = F_x - \frac{1}{\rho}\cdot\left(-\rho\frac{\partial V}{\partial x} + f'(x)\right) + \frac{\eta}{\rho}\nabla^2 u$$

$$= \frac{1}{\rho} f'(x) + \frac{\eta}{\rho}\nabla^2 u$$

$$\therefore f'(x) = \text{not a f'n of } x = A$$

$$\therefore f(x) = Ax$$

$$\therefore p = -\rho V + Ax .$$

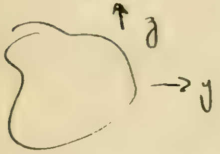
$\therefore \frac{\eta}{\rho} \nabla^2 u = \frac{\partial u}{\partial t} + \frac{A}{\rho}$

if flow is stationary, $\frac{\partial u}{\partial t} = 0$,

then $\nabla^2 u = A/\eta$. (a constant).

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = A/\eta .$$

B.C. : $u = 0$ at boundary of cross section.



if we have a circle, get Poiseuille formula:

in circular coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = A/\eta = a .$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = ar$$

$$r\frac{\partial u}{\partial r} = \frac{1}{2}ar^2 + b$$

$$\frac{\partial u}{\partial r} = \frac{1}{2}ar + b/r$$

$$u = \frac{1}{4}ar^2 + b\lg r + c$$

require also u finite everywhere.

$b \neq 0$ only if no fluid at center –

$$b = 0$$

$u = 0$ at $r = R$:

$$\therefore u = \frac{1}{4}a(r^2 - R^2) .$$

Feb. 16, '42

If flow is non stationary: suppose $u = u(y)$ flow along plate in $x \cdot z$ plane; let plate perform small oscillations then at boundary $u = A\sin\omega t$.

Equation is: $\frac{du}{dt} = \frac{\eta}{\rho}\frac{\partial^2 u}{\partial y^2}$ if no forces in x-direction

~~then $i\omega u = \frac{\eta}{\rho}$~~ if $u = u(y)e^{i\omega t}$

$$\therefore i\omega u(y) = \frac{\eta}{\rho}\frac{d^2 u(y)}{dy^2} .$$

$$\therefore u(y) = B e^{\sqrt{i\omega \rho/\eta}\, y}$$

Let $k = \sqrt{\frac{\omega\rho}{2\eta}}$. $\sqrt{2i}\, k = \pm(1+i)k$.

$$\therefore u = B e^{\pm(1+i)ky} e^{i\omega t}$$

take sol'n with neg. real part (damping)

$$= B e^{-ky} e^{i(\omega t - ky)} .$$

$$R.P. : u = B e^{-ky} \cos(\omega t - ky) .$$

(time lag ; motion of [illegible] at t, y, corresp. to motion of boundary at $t' = t - \frac{k}{\omega} y$.

$\frac{1}{k}$ gives thickness of layer affected by motion of plane; "boundary layer".

$$\frac{1}{k} = \sqrt{\frac{2\eta}{\omega\rho}} .$$

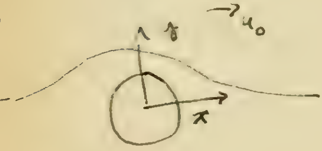
boundary layer thickness increases with η, decreases with ρ & ω . ; this is so in general.

Stokes' law

Small sphere thru fluid; R very small if defined by radius of sphere & its velocity. then if radius $\underline{a}$, $R = \frac{a u_0 \rho}{\eta} << 1$.

Neglect inertial term. No external forces. Stationary flow.

then: $\eta \nabla^2 \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \frac{dp}{d\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}}$.



near sphere get v & w as well as u.

use both x, y, z & $r\, \theta\, \varphi$ (polar axis on x).

Expect: u independent of φ.

$u(x) = u(-x)$.

$v(y) = -v(-y)$ v even in z, odd in x, odd in y.

sphere at rest

vel. at inf $= u_0$

$(v_0 = 0, w_0 = 0)$.

$w \begin{Bmatrix} \text{odd in } z \\ \text{odd in } x \\ \text{[illegible]} \end{Bmatrix}$ p vanishes at ∞ .

p indep. of φ. p odd in x (gives resultant force on sphere)

p even in y & z.

$U = V_{z0}$

$\underline{v} = 0$ on sphere.

div $\underline{v} = 0$.

$$\eta \nabla^2 \underline{v} = \nabla p.$$

$$\therefore \eta \nabla^2 \nabla \cdot v = \nabla \cdot \nabla p = \nabla^2 p = 0.$$

$\therefore$ pressure satisfies Laplace's eq'n.

$\therefore p = \frac{1}{r}$: not odd in x.

next : $p = \frac{\cos\theta}{r^2}$ OK.   [or $p = \frac{P_n}{r^{n+1}}$. ← can show not possible]

[solutions $r^n P_n$ out because p vanishes at ∞]

$$\therefore p = \frac{\cos\theta}{r^2} = A\frac{x}{r^3}.$$

$$\frac{\partial p}{\partial x} = A\left[\frac{1}{r^3} - \frac{3x}{r^4}\cdot\frac{x}{r}\right] = -A\,\frac{3x^2-1}{r^5}.$$

$$\frac{\partial p}{\partial y} = -A\,\frac{x}{r^4}\cdot\frac{y}{r} = -\frac{Axy}{r^5}$$

$$\frac{\partial p}{\partial z} = -A\,\frac{xz}{r^5}.$$

{Legendre f'ns of order 2 ÷ r^3}

$$\therefore \nabla^2 v_x = -\frac{2A}{\eta}\cdot\frac{1}{r^3}\cdot\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$$

$$= -\frac{2A}{\eta}\,\frac{P_2(\theta)}{r^3}.$$

assume $v_x = f(r)\,P_2(\theta)$.

in fact assume $v_x = \frac{\text{const}}{r} P_2(\theta)$.

$$\nabla^2 v_x(r,\theta) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r v_x) + \frac{1}{r^2}\Omega(v_x)$$

$$= -\frac{l(l+1)}{r^2} \cdot \text{const}. = -\frac{6\,\text{const.}}{r^3} P_2$$

hence $v_x = \frac{A}{3\eta r}(3\cos^2\theta - 1)$ $\therefore$ const. $= \frac{A}{3\eta}$.

similarly $v_y = \frac{A}{3\eta r}\,\frac{3xy}{r^2}$ $v_x = \frac{A}{3\eta r}\left(3\frac{x^2}{r^2}-1\right)$.

$v_z = \frac{A}{3\eta r}\,\frac{3xz}{r^2}$.

these satisfy conditions, except $\operatorname{div} \underline{v} = 0$.

∴ 1. add a term to v_x which makes $\operatorname{div} \underline{v} = 0$

2. " terms to $\underline{v}$ — they must also satisfy $\nabla^2 \underline{v} = 0$;

then if $\nabla^2 \underline{v}_2 = 0$, $\nabla\cdot\underline{v}_2 = 0$, then $\nabla\times\underline{v}_2 = 0$

$\therefore \underline{v}_2 = -\nabla\psi$.

also $\nabla^2\psi = 0$.

Feb. 20

Assumptions made so far, for Stokes law:

1. Stationary: $\frac{\partial \underline{v}}{\partial t} = 0$
2. Slow: neglect $\underline{v}\nabla\cdot\underline{v}$
3. No external forces.

With this, we got:

$$\nabla^2 p = 0$$

solution $p = A\frac{\cos\theta}{r^2}$ was considered appropriate.

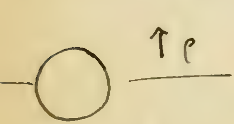
whence $\nabla^2 \underline{v} = \frac{1}{\eta}\nabla p$ can be found.

then found a particular integral:

$$\left\{\begin{aligned} v_x &= +\frac{A}{6\eta r}\left(3\frac{x^2}{r^2}-1\right)\\ v_y &= +\cdot\cdot\left(\frac{3xy}{r^2}\right)\\ v_z &= +\cdot\cdot\left(\frac{3xz}{r^2}\right)\end{aligned}\right\}$$

but these don't satisfy $\nabla\cdot\underline{v} = 0$;

hence must add solutions of $\nabla^2\underline{v} = 0$, which will make $\nabla\cdot\underline{v} = 0$.



Now $v_\rho = \frac{v_y y + v_z z}{\sqrt{y^2+z^2}} = \frac{A}{6\eta r}\cdot\frac{3x}{r^2}\sqrt{y^2+z^2}$

$= \frac{A}{6\eta r}\cdot 3\cos\theta\sin\theta$.

also $\begin{cases} v_r = v_x\cos\theta + v_\rho\sin\theta \\ v_\theta = -v_x\sin\theta + v_\rho\cos\theta \end{cases}$

whence $v_r = \frac{A}{6\eta r}\cos\theta(3\cos^2\theta - 1 + 3\sin^2\theta)$

$= \frac{A}{6\eta r}\cdot 2\cos\theta.$

$v_\theta = \frac{A}{6\eta r}\sin\theta(-3\cos^2\theta + 1 + 3\cos^2\theta)$

$= \frac{A}{6\eta r}\sin\theta.$

$\therefore \nabla\cdot\underline{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)$

$= \frac{A}{6\eta r^2}\cdot 2\cos\theta + \frac{A}{6\eta r^2}\cdot 2\cos\theta = \frac{A}{6\eta r^2}(4\cos\theta)$

$\therefore$ need to add to $\underline{v}$ a velocity field with $\nabla\cdot v' = -$ ↖

Can do this with: $v_x' = \frac{\text{const}}{r}$; $v_y = v_z = 0$

then $\nabla\cdot\underline{v}' = -\frac{\text{const}}{r^2}\cos\theta$

$\therefore \text{const} = \frac{4A}{6\eta}$

$\therefore v_x' = \frac{4A}{6\eta r}$

$\therefore$ must add to v_r: $v_x'\cos\theta$ $\quad\therefore$ New $v_r = \frac{A}{6\eta r}\cdot 6\cos\theta$

~~v_θ~~:

$= \frac{A}{\eta r}\cdot\cos\theta.$

add to v_θ: $v_x'\sin\theta$

$\therefore$ New $v_\theta = \frac{A}{6\eta r}\sin\theta[1-4]$

$= -\frac{A}{2\eta r}\sin\theta.$

Now these satisfy eq'ns. of motion & continuity condition;

Need to satisfy boundy cond.; $\therefore$ solution which we add satisfy $\nabla^2\underline{v}_2 = 0$, $\nabla\cdot\underline{v}_2 = 0$, in fact $\nabla\times\underline{v}_2 = 0$ $\quad\therefore \underline{v}_2 = -\nabla\varphi.$

this φ must give correct velocity asymptotically.

for large r: $\underline{v} = \begin{Bmatrix} u_0 \\ 0 \\ 0 \end{Bmatrix}$.

Can do this with $\varphi = -u_0 x +$ terms vanishing at $r = \infty$.

$$= \cos\theta(-u_0 r) + \cdots\cdots$$

In fact, can satisfy everything if we take also the other sol'n of Laplace's eq'n prop. to $\cos\theta$: $\varphi = \cos\theta\left[-u_0 r + B/r^2\right]$.

then $v_{2r} = -\frac{\partial\varphi}{\partial r} = -\cos\theta\left[-u_0 - \frac{2B}{r^3}\right]$.

$v_{2\theta} = -\frac{1}{r}\frac{\partial\varphi}{\partial\theta} = -\sin\theta\left[u_0 - B/r^3\right]$.

Adding these to original sol'n $\underline{v}_1$, we get:

$$v_r = v_{1r} + v_{2r} = \cos\theta\left[+u_0 + 2B/r^3 + \frac{A}{\eta r}\right]$$

$$v_\theta = v_{1\theta} + v_{2\theta} = -\sin\theta\left[u_0 - B/r^3 + \frac{A}{2\eta r}\right].$$

with A & B can satisfy the remaining conditions at sphere:

i.e. $v_r = 0 = v_\theta$ at $r = a$:

this easily done by $\frac{A}{\eta} = -3/2\, u_0 a$

$B = 1/4\, u_0 a^3$.

$$\therefore \boxed{\begin{aligned} v_r &= u_0\cos\theta\left(1 + \frac{1}{2}\frac{a^3}{r^3} - \frac{3}{2}\frac{a}{r}\right) \\ v_\theta &= -u_0\sin\theta\left(1 - \frac{1}{4}\frac{a^3}{r^3} - \frac{3}{4}\frac{a}{r}\right) \end{aligned}}.$$

If potential flow alone had been considered, would have had only one constant, hence can satisfy only one B.C. e.g. $v_r = 0$ at sphere; get flow along sphere —

Potential flow would be completely determined by this: for $\nabla^2\psi = 0$, & if $\frac{\partial\psi}{\partial n}$ is given at sphere, & at ∞, ψ is uniquely determined.

This gives $v = 0$ at front and back of sphere ("stagnation pts")

$\therefore$ .. $v_\theta = -\frac{3}{2}u_0$ at $\theta = \pi/2$, $r = a$; speeds up, as it must.

~~[illegible]~~

at surface of sphere $\frac{\partial v_r}{\partial\theta}$, $\frac{\partial v_\theta}{\partial\theta}$ are both zero;

also $\frac{\partial v_r}{\partial r} = 0$ at surface.

{this is really required by continuity eq'n; $\nabla\cdot\underline{v} = 0$, since

$\nabla\cdot v = \frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{\partial\theta} + \cdots r\cdot\theta = 0$ $\therefore$ $\frac{\partial v_r}{\partial r} = 0$.

$\therefore$ Normal ~~flow~~ velocity has vanishing normal derivative at solid. this is generally true. $\therefore$ Stresses can only contain $\frac{\partial v_r}{\partial\theta}$; this is the only one not $= 0$.

Suppose terms in P_n had been taken for solution of $\nabla^2\psi = 0$.

$r^n P_n$ are out, ~~[illegible]~~ (for $n \geq 2$) since that gives $v \sim r$ or worse at inf. . $\therefore$ only are $r^{-n-1} P_n$ $\psi \sim \frac{C}{r^{n+1}} P_n$; then can't make $\frac{\partial\psi}{\partial r} = 0$ at $r = a$ unless $C = 0$. Similarly, if higher terms are added in pressure, get another constant D; then one B.C. could be satisfied, e.g. $v_r = 0$ at $r = a$; but not any more; hence this is out. For $n = 0$, same thing; $\psi \sim A + B/r$, $v \sim B/r^2$, & can't satisfy conditions at $r = a$.

Hence only ~~terms~~ $n = 1$ occur, as we have used.

Feb. 23 Force exerted on sphere by fluid.

∴ need τ_{rx} : force in x-direction across face ⊥ r (per unit area)

we had $\tau_{rx} = \frac{x}{r}\sigma_x + \frac{y}{r}\tau_{xy} + \frac{z}{r}\tau_{xz}$.

we had: $\sigma_x = 2\eta \frac{\partial v_x}{\partial x} - p$

$\tau_{xy} = \eta\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$

$\tau_{xz} = \eta\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$

$\therefore \tau_{rx} = \eta \frac{r}{r}\cdot\left(\nabla v_x + \frac{\partial v}{\partial x}\right) - \frac{x}{r}p$.

$= \eta\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right) - \frac{x}{r}p$.

$v_x = v_r\cos\theta - v_\theta\sin\theta$

$\therefore \tau_{rx} = \eta\left(\cos\theta\frac{\partial v_r}{\partial r} - \sin\theta\frac{\partial v_\theta}{\partial r} + \frac{\partial v_r}{\partial x}\right) - \frac{x}{r}p$

at $r=a$, $\frac{\partial v_r}{\partial r} = 0 = \frac{\partial v_r}{\partial \theta}$;

$v_\theta = -u_0\sin\theta\left(1 - \frac{1}{4}\frac{a^3}{r^3} - \frac{3}{4}\frac{a}{r}\right)$

$\therefore \frac{\partial v_\theta}{\partial r} = -u_0\sin\theta\left(+\frac{3}{4}\frac{a^3}{r^4} + \frac{3}{4}\frac{a}{r^2}\right)$

$p = A\frac{\cos\theta}{r^2}$ $A = -\frac{3\eta}{2}u_0 a$

$p\big|_{r=a} = -\frac{3}{2}\eta\, u_0 a\cdot\frac{\cos\theta}{a^2}$

$\therefore \tau_{rx} = -\eta\sin\theta\frac{\partial v_\theta}{\partial r}\Big|_{r=a} - \frac{x}{r}p\Big|_{r=a}$

$= \eta\cdot\sin\theta\cdot u_0\cdot\sin\theta\cdot\frac{3}{2}\cdot\frac{1}{a} - \frac{x}{r}p\Big|$

$= \frac{3}{2}\frac{\eta u_0}{a}\sin^2\theta - p\cos\theta$

$= \frac{3}{2}\frac{\eta u_0}{a}\sin^2\theta + \frac{3}{2}\frac{\eta u_0}{a}\cos^2\theta$

$= \frac{3}{2}\frac{\eta u_0}{a}$!

Hence pressure + viscous forces give force independent of θ !

$\therefore$ Total force $= \frac{3}{2}\frac{\eta u_0}{a}\cdot 4\pi a^2$ $= 6\pi\eta u_0 a$! Stokes law

If only this force acts on sphere, it comes to rest; if it travels, say, several diameters, flow is quasi-stationary + can be treated using above.

If a uniform force acts, reach a final velocity,

given by ~~F = 6πηu₀a~~.

Suppose particles of density ρ_m are suspended in fluid of density ρ_f.

then $F = \frac{4\pi}{3} a^3 g (\rho_m - \rho_f) = 6\pi\eta u_0 a$.

and ~~termin~~ velocity $u_0 = \frac{4\pi a^3 g(\rho_m - \rho_f)}{3 \cdot 6\pi\eta a} = \frac{2}{9} a^2 (\rho_m - \rho_f)\, g/\eta$

if $g \sim 1000$, $\eta \sim 10^{-2}$ (H_2O), ρ

$$u_0 = \frac{2}{9} \cdot \frac{10^3}{10^{-2}} (\rho_m - \rho_f) a^2$$

$$= 2 \cdot 10^4 (\rho_m - \rho_f) a^2$$

if $a \sim 10^{-3}$ cm., $u_0 \sim .02$ cm/sec if $\rho_m - \rho_f \sim 1$.

How far can this be applied? $R = \frac{u_0 (2a) \rho_f}{\eta}$

$$= \frac{4}{9} a^3 \frac{\rho_f (\rho_m - \rho_f) \cdot g}{\eta^2} \ll 1$$

if $\rho_f = 1$, $\rho_m - \rho_f = 1$,

$$R = \frac{4}{9} \cdot a^3 \cdot \frac{10^3}{10^{-4}} < 1$$

$a^3 < 2 \cdot 10^{-7}$

$a < \sqrt[3]{.2} \cdot 10^{-2}$

$a < 5 \cdot 10^{-3}$

barely visible

$u_0 \sim \frac{1}{2}$ cm/sec

Air: $\rho_f \sim 10^{-3}$

$\eta = 2 \cdot 10^{-4}$

"little raindrops" $\rho_m \sim 1$

$\therefore \frac{4}{9} \cdot 25 \cdot 10^6 \cdot 10^7$

$\therefore a^3 < 10^{-7}$

damn small raindrops!

Proof that solution we had is unique.

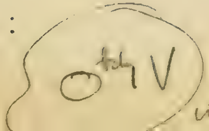
$$\eta \nabla^2 \underline{v} = \nabla p \qquad \nabla \cdot \underline{v} = 0$$

$$\therefore \underline{v} \cdot \eta \nabla^2 \underline{v} = \underline{v} \cdot \nabla p$$

$$\therefore \eta \int (v_x \nabla^2 v_x + v_y \nabla^2 v_y + v_z \nabla^2 v_z)\, d\tau = \int \underline{v} \cdot \nabla p \, d\tau$$

$$\therefore \eta \left[\int_S \left(v_x \frac{\partial v_x}{\partial n} + v_y \frac{\partial v_y}{\partial n} + v_z \frac{\partial v_z}{\partial n} \right) d\sigma - \int_V \left((\nabla v_x)^2 + (\nabla v_y)^2 + (\nabla v_z)^2 \right) d\tau \right] = \int \underline{v} \cdot \nabla p \, d\tau.$$

Consider:



infinite sphere.

at surface of body, $v_x = v_y = v_z = 0$

$\therefore$ surface integral vanishes at body

Suppose there are 2 sol'ns: $\underline{v_1}\, p_1$ $\underline{v_2}\, p_2$.

then $\underline{v} = \underline{v_1} - \underline{v_2}$ $p = p_1 - p_2$ is also a sol'n.

also $\underline{v} = 0$ at ∞ since v_1 & v_2 corresp. to same vel. etc.

then for $\underline{v}$, surface integral vanishes also at infinite sphere

(we had that a sol'n must be of form $p \sim r^n P_n$

also $\psi \sim r^n P_n$

hence velocity comp's: $v_x = v_{x_0} + \frac{v_{x_1}}{r} f(\theta) + \cdots$

but we excluded higher powers; $\therefore v_x \sim 1/r$

$\frac{\partial v}{\partial n} \sim 1/r^2$

$\therefore v \frac{\partial v}{\partial n} \sim 1/r^3$

$\therefore$ surface integral vanishes)

Likewise $\int \underline{v} \cdot \nabla p \, d\tau$

$= \int \nabla \cdot p\underline{v} \, d\tau - \int p \, \nabla v \, d\tau$

but $\nabla \cdot v = 0$; $\therefore$ $= \int_S p v_n \, d\sigma$ $= 0$ on surface of body, also on infinite sphere;

Hence $\int \left((\nabla v_x)^2 + (\nabla v_y)^2 + (\nabla v_z)^2 \right) d\tau = 0$

$\therefore \nabla v_{x_{yz}} = 0$ everywhere.

$\therefore v_x, v_y, v_z = 0$ everywhere since they vanish at ∞.

$\therefore \underline{v_1} = \underline{v_2}$

Hence solution is unique.

Feb. 25] Improvements due to Oseen. Large distances.

Can't use perturbation theory, since inertial terms which were neglected are larger there than viscous terms!

We had $u \sim u_0 (1 + \frac{a}{r} f(\theta))$.

then $\nabla^2 u \sim \frac{u_0 a}{r^3} f(\theta)$ $\quad \nabla u \sim \frac{u_0 a}{r^2} g(\theta)$.

inertial terms: $\rho(v \cdot \nabla) v \sim \rho \frac{u_0^2 a}{r^2}$

viscous ~~terms~~: $\sim \eta \frac{u_0 a}{r^3}$ $\quad$ large r, viscous < inertial.

ratio $\frac{\text{inertial}}{\text{viscous}}$: $\frac{\rho u_0^2 a}{\eta r^2 u_0 a} \cdot r^3 \sim \frac{\rho u_0 r}{\eta}$.

$\therefore$ for large enough R, our solution can't be correct

for large r, velocity is nearly u_0.

$\therefore$ approximate $\underline{v}$ by asymptotic value in inertial terms.

$(v \cdot \nabla) v = u_0 \frac{\partial}{\partial x} \underline{v}$.

then get: $\rho u_0 \frac{\partial \underline{v}}{\partial x} = -\nabla p + \eta \nabla^2 \underline{v}$

Let $\underline{v} = u_0 \underline{i} + \underline{v}'$.

divide $\underline{v}'$ into 2 parts: one eliminates pressure:

$\underline{v}' = \underline{v}_1 + \underline{v}_2$.

Assume: $\begin{cases} \rho u_0 \frac{\partial \underline{v}_1}{\partial x} = -\nabla p \\ \underline{v}_1 = -\nabla \varphi_1 \\ \nabla \cdot \underline{v}_1 = 0 \end{cases}$ (from these, $\nabla^2 \underline{v}_1 = 0$).

then $\frac{\rho u_0}{\eta} \frac{\partial \underline{v}_2}{\partial x} = \nabla^2 \underline{v}_2$

Now $\nabla^2 \varphi_1 = 0$

also $\rho u_0 \frac{\partial}{\partial x}(-\nabla \varphi_1) = -\nabla p$

$\therefore \rho u_0 \frac{\partial \varphi_1}{\partial x} = p$ (choose zero of p so that const. vanishes)

also $\nabla^2 p = 0$

$\underline{v_2}$: Let $\rho \frac{u_0}{\eta} = 2k$

if $v_2 = +\nabla \chi$, ~~then~~ and $\nabla^2 \chi = 2k \frac{\partial \chi}{\partial x}$

then $\nabla^2 \underline{v_2} = 2k \frac{\partial v_2}{\partial x}$ is satisfied;

but not $\nabla \cdot \underline{v_2} = 0$.

Now $\nabla \cdot \underline{v_2} = \nabla^2 \chi = 2k \frac{\partial \chi}{\partial x}$

$\therefore$ ~~add term~~ need to subtract something with divergence $-2k \frac{\partial \chi}{\partial x}$.

$\therefore$ Let $\underline{v_2} = \nabla \chi - 2k \chi \, \underline{i}$

then $\nabla \cdot \underline{v_2} = \nabla^2 \chi - 2k \frac{\partial \chi}{\partial x} = 0$.

Can solve by $\chi = e^{kx} \psi$

then $\frac{\partial \chi}{\partial x} = e^{kx}\left(\frac{\partial \psi}{\partial x} + k\psi\right)$.

$\frac{\partial^2 \chi}{\partial x^2} = e^{kx}\left(k^2 \psi + 2k \frac{\partial \psi}{\partial x} + \frac{\partial^2 \psi}{\partial x^2}\right)$

$\nabla^2 \chi = k^2 \chi + 2k e^{kx} \frac{\partial \psi}{\partial x} + e^{kx} \nabla^2 \psi$.

$\therefore \nabla^2 \chi - 2k \frac{\partial \chi}{\partial x} = \nabla^2 \psi - k^2 \psi = 0$

$(\nabla^2 - k^2)\psi = 0$

simplest sol'n: $\psi = \frac{1}{r} e^{\pm kr}$ throw out +.

(other sol'n's with $P_n^m(\theta, \varphi)$) $\therefore$ take $\psi = \frac{1}{r} e^{-kr}$

$\therefore \chi = \frac{A}{r} e^{-k(r-x)}$.

this sol'n has $v \cdot v = 0$ except at origin;

calculate fluid crossing sphere of radius r:

$r^2 \int \oint v_r \cdot \sin\theta \, d\theta \, d\varphi \, dr$

$v_r = \frac{\partial \chi}{\partial r} - 2k \chi \cos\theta$

make r very small: then exp ~ 1, and 2nd term is small comp. with first; $\therefore \int = -4\pi A$; i.e. a negative source (sink) at centre of sphere. Hence need to add a solution with equal sink; hence $\underline{v}_1$ must be this.

$\therefore$ choose a ϕ_1' compensating this sink: $\phi_1' = A/r$.

Can add also $\phi_1'' = \frac{Bx}{r^3} = \frac{B\cos\theta}{r^2}$, to satisfy B.C. (has right symmetry) Hence complete soln:

$$\underline{v} = u_0 \underline{i} - 2k\chi \underline{i} + \nabla\chi - \nabla\phi_1$$

$$= (u_0 - 2k\chi)\underline{i} + \nabla\left(\frac{A}{r} e^{-k(r-x)} - \frac{A}{r} - \frac{Bx}{r^3}\right)$$

On surface of sphere: (small kr) $\left[R = \frac{\rho u_0 a}{\eta} \text{ still small} \right.$ $\left. \text{i.e. } 2ka \text{ small}\right]$.

For small kr: $e^{-k(r-x)} = 1 - k(r-x)$

then $\underline{v} = (u_0 - 2k\chi)\underline{i} + \nabla\left(-\frac{Ak(r-x)}{r} - \frac{Bx}{r^3}\right)$

$\chi \sim \frac{A}{r}(1 - k(r-x))$ $\nabla\left(Ak(1-\cos\theta) - \frac{B\cos\theta}{r^2}\right)$.

Now $v_r = (u_0 - 2k\chi)\cos\theta + \frac{2B\cos\theta}{r^3}$.

$$= \left(u_0 - \frac{2kA}{r}\right)\cos\theta + \frac{2B\cos\theta}{r^3} = \cos\theta\left[u_0 - \frac{2kA}{r} + \frac{2B}{r^3}\right]$$

and $v_\theta = \left(u_0 - \frac{2kA}{r}\right)(-\sin\theta) - \frac{\sin\theta}{r}\left(Ak - \frac{B}{r^2}\right)$.

~~$= -\left(u_0 - \frac{2kA}{r}\right)\sin\theta$~~

$$= -\sin\theta\left[u_0 - \frac{kA}{r} - \frac{B}{r^3}\right]$$

at $r = a$, $v_r = v_\theta = 0$: $\therefore Ak = 3/4\, u_0 a$
$B = 1/4\, u_0 a^3$.

$\therefore$ small r sol'n same as before :

$v_r = u_0 \cos\theta\left[u_0 - \frac{3}{2}\frac{a}{r} + \frac{1}{2}\frac{a^3}{r^3}\right]$ } . same as ~~before~~ -

$v_\theta = -u_0 \sin\theta\left[1 - \frac{3}{4}\frac{a}{r} - \frac{1}{4}\frac{a^3}{r^3}\right]$.) but holds only near sphere!

this gives same resistance, in this approximation.

Can correct by a power series in ka $\quad (4ka = R)$.

then get Resistance $= 6\pi\eta a u_0 \left(1 + \frac{3}{16}R - \frac{19}{1280}R^2 + \frac{71}{20,480}R^3 \dots\right)$

Feb. 27 | We had: $\underline{v} = u_0\underline{i} + \nabla(\chi - \varphi) - 2k\chi\underline{i}$.

found $\chi = \frac{A}{r}e^{-k(r-x)}$ $\qquad k = \frac{u_0\rho}{2\eta} = \frac{R}{4a}$

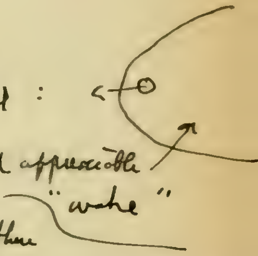
$\varphi = \frac{A}{r} + \frac{B\cos\theta}{r^2}$

$Ak = \frac{3}{4}u_0 a$, $B = \frac{1}{4}u_0 a^3$.

$\chi \to 0$ for large r except in x-direction.

χ is appreciable for, say, $k(r-x) = 1$

this is a paraboloid :



outside wake, $\underline{v} - u_0\underline{i} = -\nabla\varphi$

large r : $-\nabla\varphi = A/r^2$; source at sphere

which means that fluid ahead of body diverges -

[illegible] makes room for body. ← ⊖

acts like source of strength $A = \frac{3}{4}\frac{u_0 a}{k} = \frac{3}{4}\frac{u_0 a}{u_0\rho}2\eta = \frac{3}{2}\frac{a\eta}{\rho}$

i.e. independent of velocity of body

in wake : large distances : $r \sim x$, $\nabla(\chi - \varphi) \sim \frac{1}{r^2}$

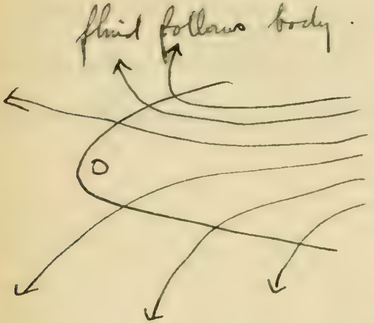
term $2k\chi$ becomes important : $\sim 1/r$.

$-2k\chi$ is most important term:

gives velocity in ~~+~~ r direction $= -2k \frac{A}{r} e^{-\kappa(r-x)}$.

$= -2 \cdot \frac{3}{4} \frac{u_0 a}{r} e^{-\kappa(r-x)}$

$= -\frac{3}{2} u_0 \frac{a}{r} e^{-\kappa(r-x)}$



pressure $= \rho u_0 \frac{\partial \varphi}{\partial x}$

$= \rho u_0 \left(\left(-\frac{x}{r^3}\right) A - \frac{B}{r^3}\left(3\cos^2\theta - 1\right) \right)$

1st term small behind body; pushes fluid out in front, sucks in fluid from behind; 2nd term is sym. about plane $x = 0$. no resultant force

---

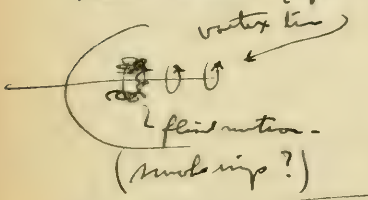
$(\nabla \times \underline{v})_y = -2k \frac{\partial \chi}{\partial z} = 2k \frac{z}{r} e^{-\kappa(r-x)} (1 + \kappa r)$ gives vortices

$(\nabla \times \underline{v})_z =$ $\quad \cdot\cdot \; \frac{y}{r}$

$(\nabla \times \underline{v})_x = 0.$ $\quad \nabla \times \underline{v}$ is $\perp$ to x; in fact circular about x

circles with x-axis as center. can draw "vortex lines"; fluid rotates about these lines.

Then ~~velocity of fluid~~ fluid rotates about these lines.



Case of cylinder has also been solved — Here we add a sol'n $\log r/a$ to the symmetric solution; but this diverges

Using Oseen's method, can get a solution like sphere — χ is a Bessel f'n of imaginary argument.;

get a constant asymptotic sol'n $\quad$ Resistance $= \dfrac{4\pi\eta\, u_0 l}{\log \frac{\eta}{\rho u_0 a} + \frac{1}{2} + \log 4 - C}$

March 6
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A.D.
?–(3)

It would be nice to use potential flow, since it is simplest; however, need to consider deviations. The greatest deviation, we found for sphere, are right around body, & in wake. So potential flow can be used in front & sides (roughly) of body, but not ~~around~~ on boundary & in wake.

Let $\underline{\omega} = \nabla \times \underline{v}$ = "vorticity".

In ~~two~~ dimensions : $v_x\ v_y\ (x, y, t)$.

Continuity : $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$

$$\omega = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$$

Eq'n of motion: $\rho \frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x} v_x + \frac{\partial v_x}{\partial y} v_y = -\frac{\partial p}{\partial x} + \eta \nabla^2 v_x$
(similar one for v_y).

Multiply 1st by $\partial/\partial y$, second by $\partial/\partial x$, & subtract:

Get : $\rho \frac{\partial \omega}{\partial t} + v_x \frac{\partial \omega}{\partial x} + v_y \frac{\partial \omega}{\partial y} = \eta \nabla^2 \omega$
(using $\nabla \cdot \underline{v} = 0$)

Eq'n for heat conduction in medium moving with velocity $\underline{v}$.
if $\eta \to$ heat conductivity, $\rho \to$ heat capacity.

$$\rho \frac{d\omega}{dt} = \eta \nabla^2 \omega .$$

Like body being hot & heating up fluid; only fluid which comes near body gets hot, i.e. part which goes into wake; high temp. means vorticity –

Suppose ω is infinite at $x = y = 0$ (in such a way that its integral over $x = y = 0$ is finite). $\omega = 0$, $x \neq 0$, $y \neq 0$. (How will this "vortex filament" spread ?

[bring a certain amt of heat ~~to~~ origin at $t = 0$, let it spread –

then $\rho \frac{\partial \omega}{\partial t} = \eta \nabla^2 \omega$;

sol'n is : $\omega = \frac{A}{t} e^{-\frac{r^2}{4\mu t}}$ $\mu = \eta/\rho$.

i.e. heat (~~vorticity~~) spreads according to this law —

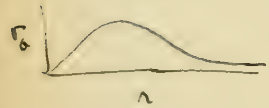
$$\int_0^a \omega \cdot 2\pi r\,dr = \int (\nabla \cdot \underline{v})_n \, d\sigma = \int_0^{2\pi} v_\theta \, r\,d\theta = 2\pi r v_\theta = \Gamma \text{ (circulation)}$$

$$\therefore \quad v_\theta = \frac{1}{2\pi r} \cdot 2\pi \int_0^a \frac{A}{t} e^{-r^2/4\mu t} r\,dr$$

$$= \frac{A}{rt} \int_0^a e^{-r^2/4\mu t} r\,dr = \frac{2\mu A}{r} \left(1 - e^{-\frac{r^2}{4\mu t}}\right)$$

if $r \gg \sqrt{4\mu t}$, $v_\theta \sim \frac{1}{r}$. i.e. Γ is constant.

if $r \ll \sqrt{4\mu t}$, $v_\theta \sim r$ i.e. $\Gamma \sim r^2$.



as t increases, vortex filament increases ;
max velocity position moves out, & value of " " decreases.

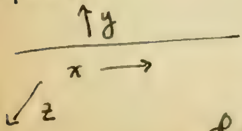
(Now we see it was OK. to neglect quadratic terms, since $(\underline{v}\cdot\nabla)\underline{v} = 0$ since grad v is radial & v is tangential.

$R = \rho \frac{u_0 l}{\eta} \gg 1$.

Stiff velocity
Fluid flowing past long flat plate

Potential flow : plate makes no difference ; doesn't satisfy $v_x = 0$ at $\eta = 0$;

Expect potential flow OK far from plate ;
Near plate, a "boundary layer" of increasing thickness with different sol'n.



Expect thickness of layer << length of plate if suff. great R.

Let δ = thickness of boundary layer ($\delta(x)$).

l = length of plate.

Continuity: $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$.

$$v_y = \text{const} - \int_0^y \frac{\partial v_x}{\partial x}\,dy$$

but $v_y = 0$ at $y = 0$, $\therefore$ const $= 0$.

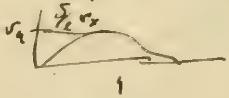
$$v_y = -\int_0^y \left(\frac{\partial v_x}{\partial x}\right) dy$$

$$\approx \frac{v_x}{l}\cdot\delta.$$

Hence in B.L.,

$$v_y \approx v_x \cdot \frac{\delta}{l}.$$

i.e.



Eq'n of motion: $\nabla^2 v_x = \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}$;

$$\approx \frac{v_x}{l^2} + \frac{v_x}{\delta^2} \approx \frac{v_x}{\delta^2}.$$

i.e. viscosity term comes from change of $\perp$ comp. of velocity,

hence neglect $\frac{\partial^2 v_x}{\partial x^2}$.

$$\rho\left[v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right]$$

$$\approx \rho\left[v_x \frac{v_x}{l} + v_x \cdot \frac{\delta}{l}\,\frac{v_x}{\delta}\right] \approx \rho\,\frac{v_x^2}{l}.$$

Assume flow doesn't change rapidly: $\rho\frac{\partial v_x}{\partial t} \lessapprox \rho\,\frac{v_x^2}{l}$.

τ = time in which flow changes by its own magnitude.

i.e. $\frac{v_x}{\tau} \lessapprox \frac{v_x^2}{l}$

i.e. $\tau \gtrapprox \frac{l}{v_x}$

| left right | i.e. time for fluid to flow length of plate.

then $\eta\,\frac{v_x}{\delta^2} \approx \rho\,\frac{v_x^2}{l}$

i.e. $\delta^2 \approx \frac{\eta\,l}{\rho\,v_x}$

i.e. $\frac{\delta}{l} \approx \sqrt{\frac{\eta}{\rho\,l\,v_x}} = \frac{1}{\sqrt{R}}$

ratio width to length of B.L. $\approx \frac{1}{\sqrt{R}}$.

From the heat (diffusion) eq'n, we know also that distance

to which heat (vorticity) goes is $x = \sqrt{4\mu t}$; $= \sqrt{\frac{4\mu x}{v_x}}$

$= \sqrt{\frac{4\eta x}{\rho v_x}}$

same as before

March 9 Eq'n of motion for y-component:

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \eta\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right)$$

↑ negligible again.

$$p(x,\delta) - p(x,0) \sim \frac{\partial p}{\partial y}\cdot\delta \sim \rho v_x \frac{v_y}{\ell}\cdot\delta \sim \rho v_x^2 \left(\frac{\delta}{\ell}\right)^2$$

∴ change in p ⊥ to surface thru B.L. may be neglected.

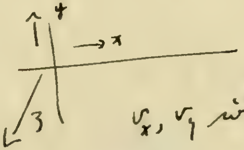
Continuity eq'n: $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$

Let $v_x = \frac{\partial \psi}{\partial y}$, $v_y = -\frac{\partial \psi}{\partial x}$; ψ : stream function.

(not a potential!).

(don't need $\nabla \times \underline{v} = 0$)

flat plate again:



$y = 0$.

v_x, v_y will depend on $\frac{y}{\delta}$ ($\delta(x)$ = width of B.L. at x)

Let $\delta(x) = \sqrt{\frac{\eta x}{\rho u_0}} = \sqrt{\mu x/u_0}$

If zero viscosity, p = const., v = const.; ∴ zero th approx: $\frac{\partial p}{\partial x} = 0$.

Assume stationary flow, $\frac{\partial v_x}{\partial t} = 0$:

then $\rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) = \eta \frac{\partial^2 v_x}{\partial y^2}$.

Introduce $\psi = \sqrt{\mu u_0 x}\, f(\xi)$ $\xi = y/\delta$

$\mu = \eta/\rho$ then $v_x = \sqrt{\mu u_0 x}\, f'(\xi)\cdot\frac{1}{\delta} = u_0 f'(\xi)$

$v_y = -\sqrt{\mu u_0 x}\, f'(\xi)\cdot -\frac{y}{\delta^2}\cdot\frac{1}{2}\frac{1}{\sqrt{x}}\cdot\sqrt{\mu/u_0} - \frac{1}{2}\sqrt{\mu u_0}\frac{1}{x^{1/2}}\cdot f(\xi)$

$= \ldots f' \frac{y\, u_0}{\ldots} - \frac{1}{2}\frac{\mu}{\ldots}f = -\frac{\sqrt{\mu u_0 x}}{\ldots}(\xi f' - f)$

Need $f'(\xi) \to 1$ for $\xi \to \infty$.

then ~~diff. eq'n becomes~~

$$\frac{\partial v_x}{\partial x} = u_0 f''(\xi) \cdot -\frac{\xi}{2x}$$

$$\frac{\partial v_x}{\partial y} = u_0 f''(\xi) \cdot \frac{\xi}{y} = u_0 f''(\xi) \cdot \sqrt{\frac{u_0}{\mu x}}$$

$$\frac{\partial^2 v_x}{\partial y^2} = u_0 \left[f'''(\xi) \cdot \frac{u_0}{\mu x} \right].$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \eta/\rho \frac{\partial^2 v_x}{\partial y^2} \quad \text{becomes:}$$

$$u_0 f'(\xi) \cdot u_0 f''(\xi) \cdot -\frac{\xi}{2x} + \frac{\sqrt{\mu u_0 x}}{2x}\left(\xi f' - f\right) \cdot u_0 f'' \sqrt{\frac{u_0}{\mu x}} = \mu \cdot u_0 f''' \frac{u_0}{\mu x}$$

or: $u_0^2 \left[f' f'' \left(-\frac{\xi}{2x}\right) + \frac{1}{2x}\left(\xi f' - f\right) f'' \right] = u_0^2 \cdot \frac{1}{x} \cdot f'''$.

$$\therefore -\xi f' f'' + \xi f' f'' - f f'' = 2 f'''$$

~~$-2\xi f' f'' + f f'' = 2f'''$~~

∴ assumption correct (about v_x)

$$\therefore f f'' + 2 f''' = 0$$

B.C.: $f' = 1$ as $\xi \to \infty$

for $\xi = 0$: $v = 0$, i.e. $f' = 0$, $f = 0$.

Can eff in series: $f = \sum \frac{A_n}{n!} \xi^n$

$$f'' = \sum \frac{A_n}{(n-2)!} \xi^{n-2} \qquad f''' = \sum \frac{A_n}{(n-3)!} \xi^{n-3}$$

$$\therefore \left(\sum \frac{A_n}{n!} \xi^n\right)\left(\sum \frac{A_n}{(n-2)!} \xi^{n-2}\right) + 2 \sum \frac{A_n}{(n-3)!} \xi^{n-3} = 0$$

Turns out $A_{1234} = 0$; $A_5 = -\frac{A_2^2}{2}$ $A_6 = 0$ $A_7 = 0$, only get A_{3n+2}.

get : $f = \sum_0^\infty C_n \frac{\alpha^{n+1}}{(3n+2)!}\left(-\frac{1}{2}\xi\right)^{3n+2}$

$c_0 = 1$
$c_1 = 1$
$c_2 = 11$
$c_3 = 375$ etc.

Converges. $\alpha = A_2$?

$c_5 \sim 4{,}000{,}000$!

For large ξ : $f_1 = \gamma\xi - \beta$ would satisfy eq'n ;

if $f = f_1 + f_2$, $f_2 \ll f_1$ for large ξ

then $f_1 f_2'' + 2f_2''' = 0$

$$\frac{f_2'''}{f_2''} = -\frac{1}{2} f_1 = -\frac{1}{2}(\gamma\xi - \beta)$$

$$\therefore \log f_2'' = ~~-\frac{1}{2}\left(\gamma\frac{\xi^2}{2} - \beta\xi\right)~~$$

$$= -\frac{1}{4}(\gamma\xi - \beta)^2$$

Can fit solution exactly by numerical methods —

$$f_2'' = \varepsilon\, e^{-\frac{1}{4}(\gamma\xi-\beta)^2}.$$

want $\gamma = 1$

March 11 $f f'' + 2 f''' = 0$

can multiply f by a, ξ by $1/a$

Hence if $f(\xi)$ is a sol'n, then $g(\xi) = a f(a\xi)$ is also.

then $g'(\xi) = a$ [illegible]

$$g g'' = a f(a\xi) + a^3 f''(a\xi)$$

$$2g''' = 2a^4 f(a\xi).$$

and $g'(\xi) = a^2 f'(a\xi)$

$\rightarrow a^2$ as $\xi \to \infty$

Numerical sol'n done by Toepfer,

$\alpha = .332$ to fit exact

sol'n at ∞.

$$f = \xi - \beta + \varepsilon\, e^{-\frac{1}{4}(\xi-\beta)^2}$$

$\beta = 1.73$

$\varepsilon = .231$

ξ	$f'(\xi)$
0	0
.4	.133
1	.330
2	.630
3	.846
4	.955
5	.9916
6	[illegible]

for $\xi \sim 5$, reach asymptotic value of velocity.

From this can calculate force on plate –

$$\text{Shear} = \tau = \eta\left(\frac{\partial v_x}{\partial y}\right). \qquad \left(\frac{\partial v_y}{\partial x} = 0\right)$$

$$\left.\frac{\partial v_x}{\partial y}\right|_{x=0} = u_0 f''(\xi)\cdot\frac{\partial \xi}{\partial y} = u_0 f''\cdot\frac{1}{\sqrt{\frac{\eta x}{\rho u_0}}} = u_0^{3/2}\cdot\sqrt{\frac{\rho}{\eta x}}\cdot\alpha.$$

$$\therefore \tau = \alpha\cdot u_0^{3/2}\cdot\sqrt{\frac{\rho\eta}{x}}.$$

Total force (per unit length) $= \int_0^l \tau\,dx = \alpha u_0^{3/2}\sqrt{\rho\eta}\cdot\left.\frac{x^{1/2}}{1/2}\right|_0^l$

$$= 2\alpha u_0^{3/2}\sqrt{\rho\eta l}.$$

Defin: Resistance coefficient C_D:

$$\text{Force} = \tfrac{1}{2}C_D\rho u_0^2 l$$

(This all for large R)

$$\text{Here } C_D = \frac{4\alpha u_0^{3/2}\sqrt{\rho\eta l}}{\rho u_0^2 l}$$

$$= 4\alpha\sqrt{\frac{\eta}{\rho u_0 l}}$$

$$= \frac{4\alpha}{\sqrt{R}} \ll 1.$$

Consider fluid crossing section from $y=0$ to $y=y_1$; $\int_0^{y_1} u\,dy$.

Compare with pot. flow: $\int_0^{y_2} u_0\,dy$

where y_2 is such that get same flow for both

then particle at y_2, before plate, goes to y_1, at plate.

Beyond $\xi \sim 5$, $y_1 - y_2$ is constant.

$$\int_0^{y_1} u\,dy = \int_0^{y_2} u_0\,dy \quad\text{i.e.}\quad \int_0^{y_1}[u_0-(u_0-u)]\,dy = \int_0^{y_2} u_0\,dy = u_0 y_2$$

$$\therefore u_0 y_1 - \int_0^{y_1}(u_0-u)\,dy = u_0 y_2$$

$$y_1 - y_2 = \frac{1}{u_0}\int_0^{y_1}(u_0-u)\,dy.$$

but if y_1 is large enough, $u_0 - u \sim 0$, hence can replace by ∞:

$$y_1 - y_2 = \frac{1}{u_0}\int_0^{\infty}(u_0-u)\,dy$$

$$= \int_0^{\infty} 1 - f'(\xi)\,d\xi \cdot \delta$$

since $f' \to \beta$, $\quad y_1 - y_2 = \delta \lim_{\zeta \to \infty} (\zeta_1 - f(\zeta_1)) = \beta \delta$.

$$\delta^* = y_1 - y_2 = \beta \delta \qquad \beta \sim 1.73$$

B.L.M. potential flow beyond $\sim 3\delta^*$.

Flow about other shape:

pot. flow: $v = 0$ at ends, v max at top & bottom;
∴ pressure max at front & back, min. at top & bottom

$p = p_0 - \frac{1}{2}\rho v^2$ $\quad$ behind, reverse flow develops in B.L.;

Hence get:

i.e. vortices form.
B.L. "separation" from rest of fluid takes place along any shape body.

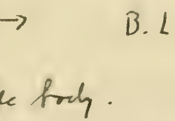
To streamline: make pressure increase slowly

At separation point $\frac{\partial v_x}{\partial y} = 0$ $\quad$ Hence $v_x \frac{\partial v_x}{\partial x}$, $v_y \frac{\partial v_x}{\partial y}$ are zero,

& get only: $\quad -\frac{\partial p}{\partial x} + \eta \frac{\partial^2 v_x}{\partial y^2} = 0$ $\quad$ from eq. of motion.

(In fact this holds on surface of body) $\quad - \quad \frac{\partial^2 v_x}{\partial y^2} = \frac{1}{\eta} \frac{\partial p}{\partial x}$. $\quad$ before S.P.

if p is decreasing, get

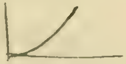


if p is increasing, get

or:

if $\frac{dp}{dx}$ is suff. l...

at separation pt.:



after S.P.

March 13 | Bernoulli theorem :

Potential flow : $\underline{v} = \nabla \varphi$ $\quad \nabla \times \underline{v} = 0$ $\quad \frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x}$ etc.

Stationary flow : $\rho (\underline{v} \cdot \nabla) \underline{v} = -\nabla p + \eta \nabla^2 \underline{v}$ but no viscosity (potential flow)

$\therefore \quad (\underline{v} \cdot \nabla) \underline{v} = -\frac{1}{\rho} \nabla p$.

$\therefore \quad \rho v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} = -\frac{\partial p}{\partial x}$

or $\rho v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial x} + v_z \frac{\partial v_z}{\partial x} = -\frac{\partial p}{\partial x}$

at a boundary, if u is tangential velocity, $p = p_0 - \frac{1}{2}\rho u^2$.

or $\rho \, \underline{v} \cdot \frac{\partial \underline{v}}{\partial x} = -\frac{\partial p}{\partial x}$

$\therefore \rho \frac{1}{2} \frac{\partial v^2}{\partial x} = -\frac{\partial p}{\partial x}$

$\therefore \frac{1}{2}\rho v^2 = -p + p_0$

or $p = p_0 - \frac{1}{2}\rho v^2$.

Physically, Bernoulli's th. says that work done by pressure goes into kinetic energy.

If curl $\underline{v} \neq 0$, this is true if you follow a certain part of fluid along a streamline. Pot. flow : p_0 same for all stream lines ;

Non-pot. flow : p_0 different for each stream line

Turbulence : volume elements are broken down, then can't use it at all.

[" starts $\sim R \sim 10^5$]

Hence can get pressure distrib. from velocity distrib. (pot flow)

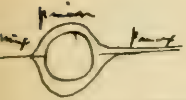
$\nabla \cdot \underline{v} = 0$; $\therefore \nabla^2 \varphi = 0$; $v_n = \frac{\partial \varphi}{\partial n} = 0$ at boundary.

$\nabla^2 \varphi = 0$, with B.C. $\frac{\partial \varphi}{\partial n}$ at walls & φ known at ∞ is often soluble ; e.g. sphere (which we did). We got $v_\theta = u_0 \left(1 + \frac{1}{2}\frac{a^3}{r^3}\right) \sin\theta$

$\varphi = u_0 \cos\theta \left(r + \frac{1}{2}\frac{a^3}{r^2}\right)$ $\quad v_r = u_0 \cos\theta \left(1 - \frac{a^3}{r^3}\right)$

In this case, stagnation pt. at $\theta = 0$, $r = a$; $p = p_0$;

at $\theta = \pi/2$, $v_\theta = 3/2\, u_0$, $\therefore p_{min} = p_0 - \frac{1}{2}\rho v_\theta^2$



Only 1st part of the distrib. has sense.

At separation point, $y = 0$, $\frac{\partial u}{\partial y} = 0$ if u is velocity in tangential direction (direction of x - coord. on body) y is coord. ⊥ to body. origin at stag. pt.

Pot. flow, $u_1 = a_1 x + \cdots$ (expand) Suppose body symmetrical about direction of incident fluid; then velocity is odd f'n ;

Hence $u_1 = a_1 x + a_3 x^3 + \cdots$; $p = p_0 - \frac{1}{2}\rho\, a_1^2 x^2 - \cdots x^4 \cdots x^6 \cdots$

and $\frac{\partial p}{\partial x} = -\rho a_1^2 x - \cdots x^3 \cdots$

$-\frac{1}{\rho}\frac{\partial p}{\partial x} = a_1^2 x + 4 a_1 a_3 x^3 + (6 a_1 a_5 + 3 a_3^2) x^5 \cdots$

Introduce $\psi(x, y)$;

$u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$; take $\psi = \psi_1(y)\cdot x + \psi_3(y)\, x^3 + \psi_5(y)\, x^5 + \cdots$

$\psi_h = 0$ at $y = 0$ $\quad \frac{\partial \psi_h}{\partial y} = 0$ at $y = 0$

$\frac{\partial \psi_h}{\partial y} = a_h$ at $y = \infty$

Can put this in eq'n of motion, with same assumptions as before, & get finally:

[Eq'n of motion practically the same if thickness of B.L. is << radius of curvature]

$x : \psi_1'^2 - \psi_1 \psi_1'' = p_1 + \mu \psi_1'''$ $\quad p_1 = a_1^2$

$x^3 : 4\psi_1'\psi_3' - 3\psi_1''\psi_3 - \psi_1 \psi_3'' = p_3 + \mu \psi_3'''$ $\quad p_3 = 4 a_1 a_3$ etc.

etc. Write $\psi_1(y) = \sqrt{a_1 \mu}\; f_1\!\left(y\sqrt{\frac{a_1}{\mu}}\right)$.

then get: $f_1'^2 - f_1 f_1'' = 1 + f_1'''$.

argument $y\sqrt{\frac{a_1}{\mu}} \sim \frac{y}{\sqrt{\frac{\mu}{a_1}\, dx}} \sim$ thickness of B.L.

this equation is integrated numerically subject to B.C.:

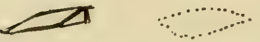
$f_1 = 0$, $f_1' = 0$ at $y = 0$; $f_1' = 1$ at $y = \infty$ [$\psi' = a_1$ at $y = \infty$].

Solution looks like one we had before;

Then get ψ_3; then $\psi = \psi_1 x + \psi_3 x^3$

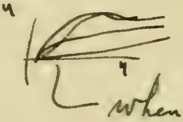
and $\left(\frac{\partial u}{\partial y}\right)_0 = \left(\frac{\partial^2 \psi}{\partial y^2}\right)_0 = a_1^{3/2} x f_1'' + a_3 \sqrt{a_1}\, x^3 f_3''$

equate to 0 to find separation pt. (a_3 is negative).

(Hiemenz) 

April 6 We had discussed separation of boundary layer by expansion in terms of x: distance from stagnation pt ; u + v, (velocity along + ⊥ to surface) expressible in terms of ~~potential~~ function ψ: $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

$\psi = x\psi_1 + x^3\psi_3 + \cdots$; $\psi_1\left(\frac{y}{\sqrt{\mu/a_1}}\right)$, a_1 from: $U_1(x) = a_1 x + a_3 x^3 + \cdots$

velocity profile changes with x: 

when $\frac{\partial u}{\partial y} = 0$ at $y = 0$, separation pt.

Behind separation pt., can't do anything by this method - have vortices & diff

Can also investigate change with time - Blasius Zs. für Ph. & M., VI 1908

Boltze - 3 dim. case -

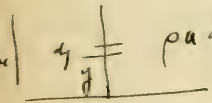
Approximate methods, easier to handle are desirable.

Kármán.

Consider flat plate again.

friction force τ measures momentum transfer from fluid to plate. $\tau = \frac{dM}{dx}$ x = coörd along plate

M = total loss of mom. since beginning.

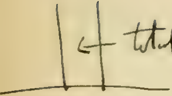


$\rho u\, dy$ crosses unit area.

$\therefore$ mom $= \rho u dy \cdot u$

loss of momentum $= \rho u dy\,(u_1 - u)$.

Total $= \int_0^{\delta} \rho u\,(u_1 - u)\,dy$ $\qquad \tau = \eta\left(\frac{\partial u}{\partial y}\right)_0$



total mom going out = mom. coming in

(Kármán: assume a reasonable distrib; then find thickness of B.L.

assume $u \neq u_1$ only if $y < \delta(x)$ assume: in B.L., $u = u(y/\delta) = u(\xi)$

Then $u = u_1 f(\xi)$ where $f(\xi) \to 1$ for $\xi \to \infty$ $\quad \therefore M = \rho u_1^2 \delta \int_0^1 f(1-f)\,d\xi$

$= \rho u_1^2 \delta \alpha$

$$\tau = \frac{\eta u_1}{\delta}\left(\frac{\partial f}{\partial \xi}\right)_0 = \frac{\eta u_1}{\delta}\beta$$

$$\therefore \tau = \rho u_1^2 \alpha \frac{d\delta}{dx} = \beta \eta \frac{u_1}{\delta}$$

Thickness of boundary layer $\sim x^{1/2}$

$$\therefore \delta \frac{d\delta}{dx} = \frac{\beta}{\alpha} \cdot \frac{\eta}{\rho u_1}$$

$$\delta^2 = \frac{2\beta}{\alpha} \frac{\eta x}{\rho u_1}$$

Need only β & α.

$$\therefore \delta = \sqrt{\frac{2\beta}{\alpha}} \sqrt{\frac{\eta x}{\rho u_1}}$$

guess a suitable f:

B.C. on f: $f = 0$ at $\xi = 0$

$f = 1$ at $\xi = 1$

try $f = \xi$ then $\alpha = \int_0^1 \xi(1-\xi)\,d\xi = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ and $\sqrt{\frac{2\beta}{\alpha}} = \sqrt{\frac{2 \cdot 1}{1/6}}$

$\beta = f' \quad = 1$ $\qquad \therefore \delta = 2\sqrt{3}\sqrt{\frac{\eta x}{\rho u_1}}$

get $\tau = \beta/\delta \cdot \eta u_1 = \frac{2 u_1}{2\sqrt{3}\sqrt{\frac{\eta x}{\rho u_1}}} = \frac{1}{2\sqrt{3}} \cdot \rho^{1/2} \eta^{1/2} u_1^{3/2} x^{-1/2}$

$$2M = \rho u_1^2 \delta\alpha = 2\rho u_1^2 \sqrt{\frac{\eta x}{\rho u_1}} \cdot 2\sqrt{3} \cdot \frac{1}{6} = \left(\frac{2}{\sqrt{3}}\right) \sqrt{\eta \rho u_1^3 x} \quad \to 1.155$$

Displacement of flow in y direction $\delta^* = \frac{\int (u_1 - u)\,dy}{u_1}$ const: 1.328.

$$= \frac{\delta u_1 \int_0^1 (1-f)\,d\zeta}{u_1}$$

$$= \tfrac{1}{2} 2\sqrt{3}\sqrt{\frac{\eta x}{\rho u_1}}$$

$$= \sqrt{3} \cdot \sqrt{\frac{\mu x}{u_1}}$$

exact theory gave $1.73 \sqrt{\mu x / u_1}$.

$\therefore$ OK.

From hydrodyn eq'ns:

$$\eta \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} + \rho\left(u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y}\right)$$

($\partial p/\partial x \to 0$)

at surface of plate, $u = v = 0$;

$\therefore f'' = 0$ at $\zeta = 0$;

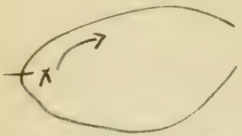
$\therefore$ take $f = \frac{3}{2}\zeta - \frac{1}{2}\zeta^3$

also want $f' = 0$ at $\zeta = 1$, for smooth join –

this gives $2M = 1.29$ – within 3%.

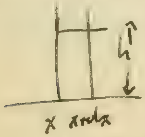
([illegible]8) Try same method on curved surface

$u_1(x)$ = velocity along x. (potential flow)



$$\tau = \frac{dM}{dx}$$

Consider section of fluid between planes x & $x + dx$



If flow is non-stationary, there is a change of mom.: $\frac{d}{dt}\left(\int_0^h \rho u\,dy\right) dx = \rho\,dx \int_0^h \frac{du}{dt}\,dy$

Net outflow of mom. thru sides $= \rho\,dx \frac{d}{dx}\left(\int_0^h u^2\,dy\right)$

[amount of fluid leaving $= \rho u\,dy$, mom. $= \rho u^2\,dy$].

(These are both supplied by plate). [h is in pot. flow region]

Amount leaving thru upper surface : $(\rho v\, dx)\cdot u_1$

Now $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$ $\therefore$ $v = -\int_0^h \frac{\partial u}{\partial x}\, dy$

Momentum supplied by external forces: pressure difference : $-h\, dx \frac{dp}{dx}$.

also some from friction : $-\tau\, dx$

from Bernoulli's eq'n : $p = \text{const.} - \frac{1}{2}\rho u_1^2$. also p changes very slightly in boundary layer. $\therefore$ external force due to pressure becomes : $h\, dx\, \rho\, u_1 \frac{du_1}{dx}$.

$$\therefore\ -\tau + h\rho u_1 \frac{du_1}{dx} = \rho\left[\int dy\, \frac{du}{dt} + \frac{d}{dx}\int u^2 dy + v u_1\right].$$

$$\downarrow$$

$$-\int \frac{\partial u}{\partial x}\, dy$$

Correction: can't use Bernoulli th. as used above, since previously assumed not stationary flow: $\therefore$ use eq'n of motion:

$$\frac{dp}{dx} = -\rho\left(\frac{\partial u_1}{\partial t} + u_1 \frac{du_1}{dx}\right)$$

$$\therefore\quad -\frac{\tau}{\rho} + h\left(u_1 \frac{du_1}{dx} + \frac{\partial u_1}{\partial t}\right) = \int_0^h\left(\frac{du}{dt} + \frac{d}{dx}u^2 - u_1\frac{d}{dx}u\right)dy$$

$$\therefore\ \frac{\tau}{\rho} = \frac{\partial}{\partial t}\int_0^h dy\,(u_1 - u) + \frac{d}{dx}\int_0^h (u_1^2 - u^2)\, dy - u_1 \frac{d}{dx}\int_0^h (u_1 - u)\, dy$$

Stationary flow: 1st integ. = 0

$$\therefore\ \frac{\tau}{\rho} = \frac{1}{\rho}\frac{dM}{dx} = \frac{d}{dx}\int(u_1^2 - u^2) - u_1\frac{d}{dx}\int(u_1 - u)\, dy$$

(for flat plate) $M = \rho \int u(u_1 - u)\, dy$

define

momentum loss

$$= \rho\left[\int (u_1^2 - u^2) - \int u_1(u_1 - u)\, dy\right]$$

$$\frac{1}{\rho}\frac{dM}{dx} = \frac{d}{dx}\int (u_1^2 - u^2) - u_1 \frac{d}{dx}\int (u_1 - u) - \frac{du_1}{dx}\int u_1 - u$$

$$= \frac{\tau}{\rho} - \frac{du_1}{dx}\int_0^h (u_1 - u)\, dy \quad = u_1 \delta^*$$

$$\therefore \quad \tau = \frac{dM}{dx} + \rho \delta^* u_1 \frac{du_1}{dx}$$

$$= \frac{dM}{dx} - \delta^* \frac{dp}{dx}.$$

Now assume: $u = ay + by^2 + cy^3 + dy^4$

eq. of motion: $\eta \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} + \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = 0$ at surface

$\frac{\partial u}{\partial t} = 0$, stat. flow.

$$\therefore \quad \frac{\partial^2 u}{\partial y^2} = \frac{1}{\eta}\frac{dp}{dx} = -\frac{1}{\eta}\rho u_1 \frac{du_1}{dx}$$

$$\therefore \quad \left.\frac{\partial^2 u}{\partial y^2}\right|_0 = b = -\frac{u_1}{\mu}\frac{du_1}{dx}$$

$$\tau = \eta \frac{\partial u}{\partial y} = \eta a$$

require $u(\delta) = u_1$

$u'(\delta) = 0$

$u''(\delta) = 0$

Define $\lambda = \frac{\delta^2}{\mu}\frac{du_1}{dx}$

then $a = \frac{u_1}{6\delta}(12 + \lambda)$, $\quad b = -u_1 \frac{\lambda}{2\delta^2} \quad c = -\frac{u_1}{2\delta^3}(4 - \lambda)$

$d = \frac{u_1}{6\delta^4}(6 - \lambda)$.

putting these in integrals above,

get finally a D.E. for δ.

$$\frac{dz}{dx} = \frac{0.8}{u_1}\left(\frac{-9072 + 1670.4\lambda - \lambda^2\left(47.4 + 4.8\frac{u_1 u_1''}{u_1'^2}\right) - \lambda^3\left(1 + \frac{u_1 u_1''}{u_1'^2}\right)}{(\lambda - 12)(\lambda + 17.76)}\right).$$

$$z = \frac{\delta^2}{\mu} = \frac{\lambda}{\left(\frac{du_1}{dx}\right)} \qquad = f(z, x).$$

f is known as soon as we know

u_1 - i.e. pot. flow. solution.

at $u_1 = 0$, forward stag. pt., $\frac{dz}{dx}$ must not blow up, ∴

equate numerator to zero & get $\lambda = 7.052$

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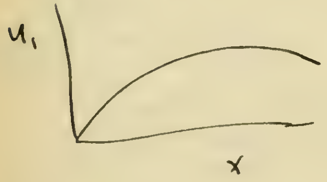
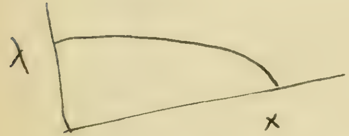
~~For small x, we always put before~~, $\delta \sim \delta_x$, ~~i.e. $\delta^2 \sim x$~~

Integrate above D.E. subject to $\lambda = 7.052$ at $x = 0$.



since δ^2 & μ are both > 0, λ has same sign as $\frac{du_1}{dx}$;

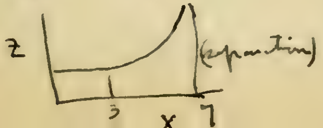


$\lambda = 0$ when $\frac{du_1}{dx} = 0$, i.e. when u_1 is a max., i.e. when pressure is min.

$\frac{du}{dy} \sim a = \frac{u_1}{6\delta}(12 + \lambda) = 0$ at $\lambda = -12$

∴ separation pt. at $\lambda = -12$

Integration gives:



cylinder : $u_1 = 7.15x - .645x^3 - .00033x^5$

separation pt. agrees with [?] result from exact calculation —

Nossns

ΠΡΟΠCT

suppose we want $\lambda = -\bar{\lambda}$

then $z = -\frac{\bar{\lambda}}{u_1}$

$$z' = \frac{\bar{\lambda}\, u_1''}{u_1'^2}$$

from D.E., get $\frac{\bar{\lambda}\, u_1 u_1''}{u_1'^2} = F\left(\bar{\lambda}, \frac{u_1 u_1''}{u_1'^2}\right)$.

$\bar{\lambda} = -12$,	$\frac{u_1 u_1''}{u_1'^2} = \infty$
$\bar{\lambda} = 10$	11.13
12	11.09

$\therefore$ [crossed out]

$$\therefore \frac{u_1 u_1''}{u_1'^2} = G(\bar{\lambda}).$$

if λ is constant, then $\frac{u_1 u_1''}{u_1'^2} = \text{const.} = \sigma$.

$$\therefore \frac{u_1''}{u_1'} = \sigma \frac{u_1'}{u_1}$$

$$\therefore u_1' = C_1 u_1^{\sigma}.$$

[crossed out: $u_1'' = C\sigma u_1^{\sigma-1} u_1'$, $\frac{u_1''}{u_1'} = \sigma \frac{u_1'}{u_1}$]

Can prevent separation only if pressure change is very slow:

$$\therefore \frac{u_1'}{u_1^{\sigma}} = C_1$$

$$\therefore u_1^{-\sigma+1} = (-\sigma+1)\cdot C_1 x + C_2$$

if $\sigma \sim 11$, $u_1^{-10} = 10Cx + C_2$

$$u_1 = \frac{u_0}{(1+10Cx)^{\frac{1}{10}}}.$$

increase of velocity is slow;
because decrease of pressure low.

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$$u_1 = \frac{u_0 c}{(1+10cx)^{.1}}$$

$$\therefore u_1' = -.1\frac{u_0 \cdot 10c}{(1+10cx)^{1.1}} = \frac{-u_0 c}{(1+10cx)^{1.1}}$$

$$\therefore \frac{\delta^2}{\mu} = \frac{\lambda}{u_1'} = \frac{-10}{-\frac{u_0 c}{(\ \ )}} = \frac{10}{u_0 c}(1+10cx)^{1.1}$$

$$\therefore \delta = \delta_0(1+10cx)^{.55}$$

Lots of exp to show how long a stream lined body should be, to get a

~~lots more exp on various ways to help streamlining~~ —

We have been ~~talking~~ always about laminar flow; this only really happens at low velocities. At higher velocities, get turbulence, which is a Bad Thing. Practically: need more pressure to get a given amt of fluid thru a pipe. Theoretically: because nothing is known about it. A given tube of stuff (say red coloring) doesn't stay together, but breaks up; And this can be a Good Thing, because it churns up the boundary layer, & helps prevent separation effect. For airplanes, have ~~turbulent~~ flow from a few cm. beyond edge of wing.

Reynolds — Turbulence starts at same value of R (Reynolds found this with expts on lots of pipes, with different fluids & velocity

Turbulence starts at $R \sim 3000$ if $R = \frac{\bar u d}{\mu}$ — $\bar u$ av. vel., d diam. of tube.

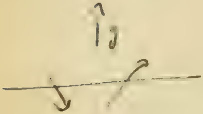
if water: $\mu = .01$

if $d = 1$ cm, $\bar u \sim \frac{3000 \cdot .01}{1} = 30$; moderate velocity

if mean velocities are $\bar u \neq 0$, $\bar v = 0$, $\bar w = 0$, $\bar u(y)$

there is no net transfer in $y + z$ direction, only in x;

But $\begin{matrix} u \\ v \\ w \end{matrix} = \bar v + v'$ — if it happens that particles crossing up have average $u' > 0$, crossing down have $u' < 0$, ...



this amounts to a transport of momentum across the boundary per unit time, which gives force.

mass transport is $\rho v'$; x-momentum transport is $(\rho v')u'$;

average is $\overline{\rho v' u'}$. i.e. $\tau_{xy} = -\overline{\rho u' v'} = A \frac{d\bar u}{dy}$

which is > 0

viscous case, we had $\tau_{xy} = \eta \frac{\partial u}{\partial y}$.

here we have a variable η, depends on velocity distribution in its neighborhood.

Introduce: $v^* = \sqrt{|\overline{u'v'}|} = \sqrt{\frac{\tau}{\rho}}$

We will study experiments on turbulent flow in pipes, + deduce therefrom some theory — (turbulent B. L. on flat plate).

April 15

To study ~~turbulent~~ flow, must go back to data —
We found before, $\tau = \frac{p_1 - p_2}{L} \cdot \frac{y}{2}$ for any shear in cylindrical case.
(y = dist. from center, $p_1 - p_2$ = pressure diff. betw. ends of pipe.)
This is an eq'n of equilibrium — says nothing about origin of forces, or relation between them. Before, we assumed the viscous relation $\tau = \eta \frac{\partial u}{\partial y}$ which led to Poiseuille's formula.

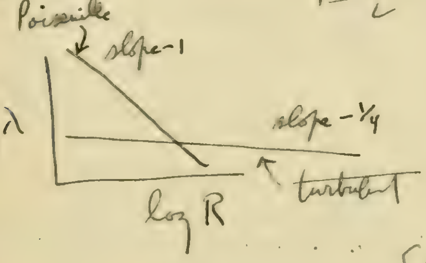
For ~~turbulent~~ case, we will take experimental relation between pressure gradient & average velocity.

~~Prandtl gives~~ We may write $\frac{p_1 - p_2}{L} = \lambda \cdot \frac{\frac{1}{2}\rho \bar{u}^2}{d}$ so that λ is dimensionless.

Observations give $\lambda = 0.316\, R^{-\frac{1}{4}}$ $\qquad R = \frac{\bar{u}\, d}{\nu}$.

If Poiseuille's law is written in same form, it gives:

$$\frac{p_1 - p_2}{L} = \frac{8\eta}{r^2}\bar{u} = \left(\frac{64}{R}\right) \cdot \frac{\frac{1}{2}\rho \bar{u}^2}{d}.$$



transition between laminar ~~& turbulent~~ flow happens at about $R = 2000$, which is a little beyond where they cross

[$R = 2{,}000$, $\lambda_P = .03$ $\quad \lambda_{turb.} = .045$]

With careful exp'ts, can make laminar flow persist, even up to $\sim 30{,}000 = R$; then need much lower (10 times so) pressures to maintain same flow.

Let us calculate v_* at $y = r$:

$$\tau_{(\text{near wall})} = \frac{p_1 - p_2}{L} \cdot r/2$$

$$= \frac{1}{2}\rho \bar{u}^2 \cdot \frac{\lambda}{d} \cdot r/2$$

$$= \frac{\lambda}{8} \cdot \rho \bar{u}^2 = \rho v_*^2 \quad \text{by definit}$$

$$\therefore v_*^2 = \lambda/8 \; \bar{u}^2 .$$

$$= \frac{.316}{8}\left(\frac{\bar{u} d}{\mu}\right)^{-1/4} \bar{u}^2$$

$$= \frac{.316}{8}\left(\frac{d}{\mu}\right)^{-1/4} \bar{u}^{7/4} .$$

$$\frac{\bar{u}}{v_*} = 7\left(\frac{v_* r}{\mu}\right)^{1/7} .$$

Can also find that in middle of tube:

vel. in middle: $u_1 = 5/4 \; \bar{u}$

$$\therefore \frac{u_1}{v_*} = 8.74\left(\frac{v_* r}{\mu}\right)^{1/7} .$$

Assume that in general, at a distance y from a wall, $\frac{u(y)}{v_*} = f\left(\frac{v_* y}{\mu}\right)$

$u(y)$ = mean velocity at y

In fact, assume same relation as found ↗ : $\frac{u}{v_*} = 8.74\left(\frac{v_* y}{\mu}\right)^{1/7}$.

this gives velocity distrib:

instead of parabola

Now this is O.K., since velocity changes only near wall, where τ is constant, & hence v_* nearly constant.

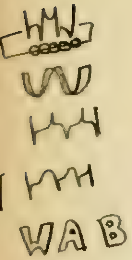
this 1/7 power law has been verified even to middle of tube, (why, nobody knows). If $u = \left(\frac{r-y}{r}\right)^{1/7} u_1$

$$\text{then } \bar{u} = \frac{\int y\,dy \left(\frac{r-y}{r}\right)^{1/7}}{r^2/2} u_1 \sim 4/5 \; u_1$$

this holds up to $R \sim 100{,}000$.

For large R, effect becomes smaller.

in general $\frac{u}{v_*} = f\left(\frac{v_* y}{\mu}\right)$ Nikuradse



in general $\frac{u}{v_*} \sim \begin{matrix} 10 \\ 30 \end{matrix}$ $\qquad \frac{v_* y}{\mu} \sim 10$, $\quad$ " $\sim 10^5$

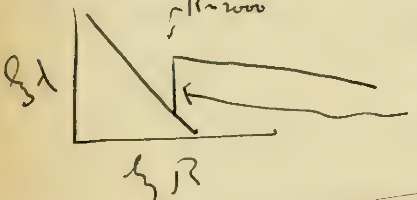
so random velocity $\sim \frac{1}{10}$ mean velocity.

$$\lambda = \frac{0.24}{\left(\log \frac{R\sqrt{\lambda}}{2.4}\right)^2}$$

for $R \sim 10^{2.5}$, $\lambda \sim .01$; doesn't change much.

For large flow, pressure gradient $\sim v^2$, low velocity, $\sim v$.

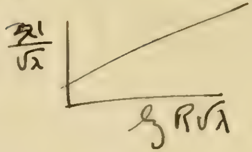
April 17 Correction: the two lines on log R, log R graph should not have been connected; should not have smoothed out Blasius law to meet Poiseuille.



discontinuous change from laminar to turbulent flow.

From experiments: $\frac{u}{v_*} = 2.5 \ln\left(\frac{v_* y}{\mu}\right) + 5.5$

$\lambda = \frac{0.24}{\left[\log_{10}\left(\frac{R\sqrt{\lambda}}{2.9}\right)\right]^2}$ (Nikuradse)



$v_* = \sqrt{\frac{\tau_{wall}}{\rho}}$

$R = \frac{\bar{u} d}{\mu}$

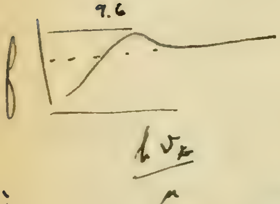
for large velocities, get $\frac{u}{v_*} = 2.5 \ln y/h + 8.5$

$h \sim$ roughness of wall —

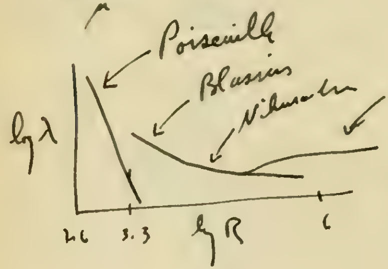
intermediate: $\frac{u}{v_*} = 2.5 \ln y/h + f\left(\frac{h v_*}{\mu}\right)$

$f = 8.5$ for large $\frac{h v_*}{\mu}$,

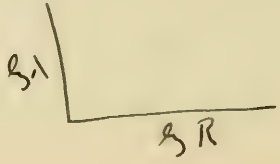
$f = 5.5 + \ln \frac{h v_*}{\mu}$ for small:



"Roughness function"



roughness of wall effect — (constant asymptotic value, higher for rougher wall.

Expt. curves: 

Flow along plate.

potential flow u_1 beyond B.L. transition between surface + pot. flow will be described by formulas similar to those for tubes.

Blasius law: $\frac{u}{v_*} = 8.74 \left(\frac{v_* y}{\mu}\right)^{1/7}$

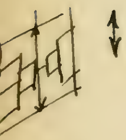
Use in Karman method; Blasius law correct in B.L. $y < \delta$.

$u = u_1 \quad y > \delta$.

Momentum loss: $M = \rho \int_0^\delta u(u_1 - u)\, dy$

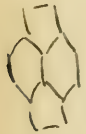
Write Blasius law as: $\frac{u}{u_1} = \left(\frac{y}{\delta}\right)^{1/7}$ which joins smoothly to $u = u_1$ at δ.

$y/\delta = \xi$



$$M = \rho \int_0^1 u_1 \xi^{1/7}(u_1 - u_1 \xi^{1/7})\, \delta\, d\xi$$

$$= \rho \delta u_1^2 \int_0^1 (\xi^{1/7} - \xi^{2/7})\, d\xi = \rho \delta u_1^2 \left[~~\frac{\xi^{8/7}}{8/7} - \frac{\xi^{9/7}}{9/7}~~ \right]$$



$$= \rho \delta u_1^2 \left[7/8 - 7/9 \right] = \frac{7}{72} \rho \delta u_1^2$$

$$\therefore \tau = \frac{dM}{dx} = \frac{7}{72} \rho u_1^2 \frac{d\delta}{dx}$$

$$= .0225\, \rho u_1^{7/4} (\mu/\delta)^{1/4}$$

$$\therefore \delta = .37 (\mu/u_1)^{1/5} x^{4/5}$$

$$\delta/x = .37\, R^{-1/5} \qquad R = \frac{\rho u_1 x}{\mu}$$

Expect turbulent flow when ~~δ/x~~

Laminar: $\delta/x = 4 R^{-1/2}$.

$$4R^{1/2} = R\, \delta/x = \frac{u_1 \delta}{\nu} \sim 2{,}000$$

$$R \sim 250{,}000$$

Exp: $R \sim 500{,}000$ for transition to turbulence.

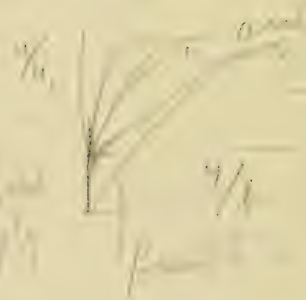
Total drag = mom. loss $= M = .036 \rho u_1^2 x R^{-1/5} = C_D \frac{1}{2} \rho u_1^2 \cdot x$ whence $C_D = .072$

laminar $C_D = 1.33$

April 10 Apply tube results to flat plate flow.

R > 500,000 (turb.)

R < 500,000 (lamin.)



Prandtl Reynolds [illegible] $\Gamma_1 = \frac{u_1 l}{\nu}$ [illegible]

[illegible] Transport that not profile depends on Γ

$\Gamma = \frac{\delta}{u_1} \frac{du_1}{dx} l^2$

[illegible] solutions [illegible] momentum theorem [illegible]

[illegible] general [illegible] $= \frac{dU}{dx} + [illegible] \frac{du_1}{dx} \delta^*$

(δ^* = displacement thickness)

[illegible]

$= [illegible] \frac{d}{dx} (\delta^* [illegible]) [illegible] \frac{dU}{dx}$

[illegible] $\frac{\delta^*}{\delta}$ [illegible] function of Γ

[illegible] $\frac{\delta^2}{\nu} \frac{du}{dx}$ [illegible]

[illegible] $\left(\frac{u_1 \delta}{\nu}\right)$ [illegible]

[illegible] Γ [illegible]

Then [illegible]

$\delta - \Gamma_1 \left(\frac{1}{\delta} - \frac{\delta^*}{[illegible]}\right) [illegible] \frac{d}{dx}\left([illegible] \left(\frac{[illegible]}{\nu}\right)\right)$

[illegible] Γ [illegible] Γ_1

[illegible]

April 22 '42

potential flow velocity $u_1 = \frac{u_0}{(1+\frac{x}{x_0})^b}$.

$b \sim 5$ from experiments. (~~turbulent flow~~).

($b \sim 10$ for laminar flow)

$Z = \delta^{**}\left(\frac{u_1\delta^{**}}{\mu}\right)^{1/4}$; $\Gamma_1 = Z\frac{u_1'}{u_1}$.

above equation obtained by requiring Γ_1 be constant such that no separation occurs. ; Separation occurs for .07, ∴ take $\Gamma \sim .05$.

[laminar flow : we had $Z = \lambda/u_1'$; took $\lambda \sim -10$] .

$$u_1'(x=0) = -\frac{u_0}{b x_0} = \frac{\Gamma_1 u_0}{z} .$$

$$Z = \frac{a}{u_1^b}\left(\int_1 u_1^b dx + c\right)$$ if take integral from front

stagnation pt., $c = 0$: $z_0 = \frac{a}{u_1^b}\int_{stag.pt.}^{0} u_1^b dx$

$$\therefore x_0 = -\frac{z_0}{b\Gamma_1} = -\frac{a\int^0 u_1^b dx}{u_1^b \cdot b\,\Gamma_1}$$

but $\frac{\int_{st}^{0} u_1^b dx}{u_1^b}$ is of order of arc length from st. to separation pt.
in fact smaller in about ratio $\frac{1}{b}$.

[For laminar flow, we had: $x_0 = \frac{u_0}{u_1'(stag)}\;\frac{K\lambda_0}{-\lambda}\cdot\frac{1}{b}$]

$\frac{K\lambda_0}{-\lambda}$?

$b \to 10$

↓ ↓

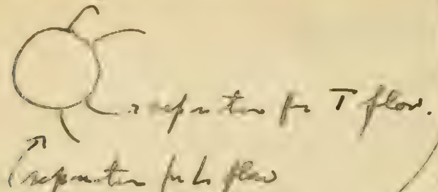
$\frac{1}{2}$ arc 2

$\therefore x_0 \sim \frac{1}{10}$ arc.

Turbulent case : $-\frac{a}{\Gamma_1} \sim \frac{1}{3}$; $x_0 \sim \frac{1}{100}$ arc.

Therefore can make body shorter without getting separation

At high reynolds numbers



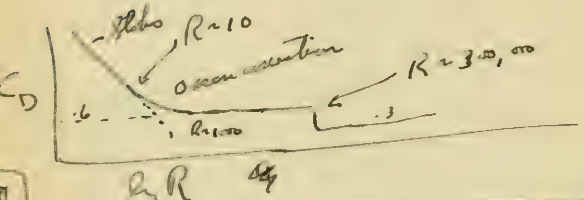
Resistance is smaller the further back the separation takes place; Hence resistance decreases (discontinuously?) when turbulence sets in, because of shift of separation pt.

Drag coefficient of sphere as fn of velocity:

$$\text{Resistance} = C_D \left(\tfrac{1}{2}\rho u_0^2\right)(\pi r^2)$$

Stokes: $C_D \sim \frac{1}{u}$ $\log C_0 \sim -\log R$

Eiffel



Flow in wake can be investigated; Wake spreads as distance from body in 2-dim. problem; as $^3\sqrt{\ }$ if a body of revolution

Back to potential flow

We know it isn't good, but after having discussed the real flow, can judge how good it is. Can be solved much easier.

Lift of an airfoil:

Impose condition that fluid can't flow around sharp edge.

(We found separation even for moderate curvatures).

With good streamlining, no separation until end;

hence potential flow solution is useful

Velocity potential φ can be introduced:

$$\underline{v} = \nabla \varphi$$

O.K. if $\nabla \times \underline{v} = 0$ everywhere

no vortices.

Bernoulli eq'n:

replace ~~$(\underline{u} \cdot \nabla) \underline{u}$ by u~~

$\underline{v} \cdot \nabla u$ by $v \cdot \frac{\partial \underline{v}}{\partial x}$ in Euler eq'n

O.K. for laminar flow; but too bad even for potential flow, except in B.L.

& get $p = p_0 - \frac{1}{2} \rho \underline{v}^2$.

Boundary Condition: $v_{normal} = 0$ [can't satisfy $v_{tan} = 0$.]

i.e. $\frac{\partial \varphi}{\partial n} = 0$.

~~Continuity~~: $\nabla \cdot \underline{v} = 0$ whence $\nabla^2 \varphi = 0$.

So need only solve Laplace's eq'n subject to $\frac{\partial \varphi}{\partial n} = 0$

on surfaces, & v_∞ = at infinity given.

Soluble for some 3-dim. cases. (e.g. sphere).
Most results in 2-dim. Good approx., for airfoils.

---

if $\varphi = \varphi(x+iy)$ then $\nabla^2\varphi = 0$.

Or: $w(x+iy) = \varphi + i\psi$; $\nabla^2\varphi = 0$ and $\nabla^2\psi = 0$.

and $\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}$, $\frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}$ (Cauchy Riemann)

$= u$, $= v$

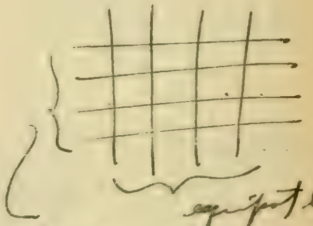
ψ = constant on a streamline — (stream function).

[$\frac{dy}{dx} = \frac{v}{u} = -\frac{\partial\psi/\partial x}{\partial\psi/\partial y}$

Consider: $w = z$.

then $\varphi = x$ $\psi = y$:

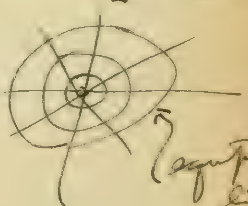
$u = 1$
$v = 0$



$w = \Gamma \log z$ $z = re^{i\alpha}$

$\varphi = \Gamma \log r$ $\psi = \Gamma\alpha$

$v_r = \frac{\partial\varphi}{\partial r} = \Gamma/r$. source at center. of strength Γ.



"Yield" of source $= \int_0^{2\pi} r\,\Gamma/r \; d\alpha$
$= 2\pi\Gamma$.

If $\Gamma = i\,\Gamma'$, $\quad \varphi = -\Gamma'\,\alpha$, $\quad \psi = \Gamma' \log r$.

$v_\theta = \frac{1}{r}\frac{\partial \varphi}{\partial \theta} = -\frac{1}{r}\Gamma'$

stream lines around circle, equipot. radial.

define Circulation $= \oint \underline{v} \cdot d\underline{s}$ around stream line

$$= \int_0^{2\pi} -\frac{1}{r}\Gamma' \cdot r\,d\theta = -2\pi\,\Gamma'.$$

to any potential flow, (e.g. for cylinder), can add a circulation, without changing Boundary condition: it will still leave flow normal to body, & will still have prescribed value at ∞.

i.e. solution in 2 dim. is not unique: therefore must add a condition — which we did: no flow around a trailing edge.

April 24 42

$w = \varphi + i\psi$; $\quad u = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}$, $\quad v = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$.

$\psi =$ constant: stream lines
$\varphi =$ constant: equi pot. lines.

Also: $\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv$ "complex velocity".

~~$-i\frac{\partial w}{\partial y} = -i\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} =$~~

If $w = z\,e^{i\alpha}$, $\frac{dw}{dz} = e^{i\alpha}$ = flow at angle α with x-axis.

$w = \frac{\Gamma}{2\pi}\log z$ $\quad$ Γ real: source of ~~strength~~ Γ.

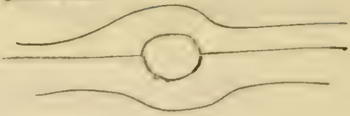
Γ imaginary: circulation Γ'.

$(\Gamma = -i\,\Gamma')$

Flow around cylinder: cylinder surface a streamline.

Suppose v_∞ = const. in x direction.

one streamline is:

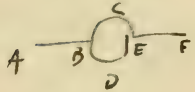


Required function is $w = z + z^{-1}$.

if $z > 1$, real, $w > 0$, real;

$z < -1$, real, $w < 0$, real;

on unit circle $z = e^{i\beta}$, $w = 2\cos\beta$, real;

∴ 

in z-plane = real axis in w-plane.

B: $z = -1$, $w = -2$ E, $z = 1$, $w = 2$ C, $z = i$, $w = 0$ D $z = -i$, $w = 0$

Can add now a circulation $-\frac{i\Gamma}{2\pi}\log z$ + still have a solution.

Most important application is to theory of airfoils.

$z = z(t)$; $w = t + \frac{1}{t}$.

$z(t)$ transforms circle into some other contour. (e.g. airfoil).

$z = t + \frac{1}{t}$ transforms unit circle in t-plane to

st. line $-2 < z < 2$ in z-plane

Displace circle a little:

t-plane.

What happens in z-plane?

Turns out ~~that~~ both sides map into a circular arc:

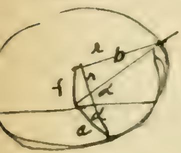
top of circle: $t = (\sqrt{1+f^2} + f)i$

$$z = (\quad)i - \frac{i}{(\;)} = i\left[\frac{(\;)^2 - 1}{(\;)}\right]$$

$$= i\left[\frac{1+f^2+2f\sqrt{\;}+f^2-1}{\sqrt{\;}+f}\right]$$

$$= 2if\,\frac{f+\sqrt{\;}}{\sqrt{\;}+f} = 2if.$$

To show upper & lower part of circle go into same arc:



$$r^2 = f^2 + a^2 - 2af\cos(90°+\alpha)$$
$$= f^2 + a^2 + 2af\sin\alpha$$

also $r^2 = f^2 + b^2 - 2bf\sin\alpha$

$$r^2 - f^2 = 1 = a^2 + 2af\sin\alpha$$
$$= b^2 - 2bf\sin\alpha$$

$c = f\sin\alpha$

whence $a = -c + \sqrt{1+c^2}$
$b = c + \sqrt{1+c^2}$.

whence $ab = 1$

a: $t = ae^{i\alpha}$

b: $t' = be^{i\alpha} = \frac{1}{a}e^{-i\alpha} = \frac{1}{t}$.   Q.E.D.

and $t + \frac{1}{t} = t' + \frac{1}{t'}$.

z the same for a & b.

Turns out that z-plane arc is circular. & till not an airplane wing.

Consider a circle tangent to above circle at $t = +1$, but larger radius:

This gives:

Kutta. Joukowski. almost a wing!

Can get different shapes by taking different values of
f & radius of circle. "Kutta - Joukowski profiles"

Take case of circular arc: get from circle with center at if,
radius $\sqrt{1+f^2}$. Flow around this circle:

$$w = \left[(t - if) + \frac{1+f^2}{t - if}\right] u_\infty \leftarrow \text{to take care of edge of radius}$$

$$- \frac{i\Gamma}{2\pi} \log(t - if).$$

Γ is undetermined;

$$\frac{dw}{dz} = \frac{dw}{dt}\frac{dt}{dz} = \frac{\frac{dw}{dt}}{\left(1 - \frac{1}{t^2}\right)}$$ velocity $= \infty$ at $t = \pm 1$ unless $\frac{dw}{dt} = 0$ there.

this would mean a flow around the trailing edge
which cannot be. ∴ adjust w so that $\frac{dw}{dt} = 0$ at $t = \pm 1$
[for wing of finite thickness $t = -1$ is not on wing - no difficulty]
[velocity at leading edge may be quite large for thin wing].
this condition fixes Γ. $\frac{dw}{dt} = u_\infty\left(1 - \frac{1+f^2}{(t-if)^2}\right) - \frac{i\Gamma}{2\pi(t-if)}$.

at $t = 1$: $u_\infty\left(1 - \frac{1+f^2}{(1-if)^2}\right) - \frac{i\Gamma}{2\pi(1-if)} = 0$

$$\frac{\Gamma}{u_\infty} = -2\pi i(1-if)\left(\frac{1-f^2-2if-1-f^2}{(1-if)^2}\right)$$

$$= -2\pi i \frac{-2f^2-2if}{1-if}$$

$$= 4\pi i f \frac{f+i}{1-if}$$

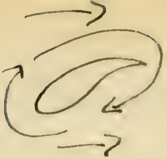
$$= -4\pi f u_\infty$$

Suppose u_∞ makes angle α
with x-axis: Simply replace
~~$(t-if)e^{-i\alpha} + \frac{1+f^2}{}$~~
$(t - if)$ by $(t - if)e^{-i\alpha}$.
(rotate thru α).

If solve for Γ this way, get

$\Gamma = -4\pi(\sin\alpha + f\cos\alpha)v_\infty$

∴ circulation is clockwise.



[this means that velocity is greater <u>above</u> than below; hence get a force up.
[in fact, in general get no force without a circulation].

For shape: get no lift for $\alpha = 0$.

But for shape → get lift even for $\alpha = 0$;
[normally adjusted to be = weight of plane at cruising speed]

April 27'42

If a is length of wing, in the general:

$\Gamma = -a\pi(f\cos\alpha + \sin\alpha)|v_\infty|$. (a instead of 4 is only difference)
(a = width of wing)

Potential flow, 2 dim:

$$\underline{F} = -l\int p\, d\underline{s}$$

$$F_y + iF_x = -l\left[\int [p\cos(n,y) + ip\cos(n,x)]\, ds\right.$$

or $F_y + iF_x$ per unit length $= \int p(dx - i\,dy)$

but $p = p_0 - \frac{1}{2}\rho v^2$

$\cos(n,y) = -\frac{dx}{ds}$

$\cos(n,x) = +\frac{dy}{ds}$

$$\therefore F_y + iF_x = -\frac{1}{2}\rho\int |v|^2(dx - i\,dy)$$

but $|v| = \frac{dw}{dz}$

$$|v|^2 = \frac{dw}{dz}\left(\frac{dw}{dz}\right)^* =$$

$$\therefore \bar{F}_y + i\bar{F}_x = -\tfrac{1}{2}\rho \oint \frac{dw}{dz}\,\frac{dw^*}{dz^*}\,dz^*$$

$$= -\tfrac{1}{2}\rho \oint \frac{dw}{dz}\,dw^*.$$

$$w = \varphi + i\psi.$$

Since contour of body is a streamline, i.e. ψ = const. on surface of body. $\therefore$ dw is real $= d\varphi$

Whence $dw^* = dw$ and

$$\bar{F}_y + i\bar{F}_x = -\tfrac{1}{2}\rho \oint \frac{dw}{dz}\,dw$$

(Blasius)

$$= -\tfrac{1}{2}\rho \oint \left(\frac{dw}{dz}\right)^2 dz$$

w at inf. $= |v_\infty| e^{-i\alpha} z$ + lower powers of z.

$$\left[\frac{dw}{dz} = |v_\infty| e^{-i\alpha} = u - iv.\right.$$

Body can be put around origin; then expand w in Laurent series around origin; $w = c_0 z + c_1 \log z + \sum \frac{c_{n+1}}{z_n}$

$$\frac{dw}{dz} = c_0 + \frac{c_1}{z_1} + \sum \frac{c_n}{z_n} \cdots$$

Hence $\oint \left(\frac{dw}{dz}\right)^2 dz = 2\pi i \left[\text{Res}\left(\frac{dw}{dz}\right)^2 \text{at } z=0\right] = 2\pi i \cdot 2c_0c_1.$

$$= 4\pi i\, c_0 c_1$$

But c_1 term is circulation term — in fact

$$c_1 = -\frac{i\Gamma}{2\pi} \quad \text{and} \quad c_0 = |v_\infty| e^{-i\alpha} = \{u_\infty\} - i\{v_\infty\}$$

$$\therefore -\tfrac{1}{2}\rho \oint = 4\pi i \cdot (u_\infty - i v_\infty)\left(\frac{-i\Gamma}{2\pi}\right) \cdot -\rho/_{\!\!\not{}}$$

$$= -\rho\Gamma(u_\infty - i v_\infty)$$

$$\therefore \boxed{\begin{aligned} F_x &= -\rho\Gamma u_\infty \\ F_y &= \rho\Gamma v_\infty \end{aligned}}$$

i.e. if $\Gamma = 0$, no force! Total force $\sim |v_\infty|$, direction $\perp$ to it. $[\underline{F} \cdot \underline{v} = 0]$. Hence doesn't do any work, no resistance associated with it. $|\text{Lift}| = \rho\Gamma|v_\infty|$

if v_∞ is $+x$ direction $\longrightarrow$, then $F_y \uparrow$ (since Γ is negative). Put Γ in:

$$F_y = \rho\pi a|v_\infty|^2(\beta\cos\alpha + \sin\alpha)$$

Numbers: ρ (air) $\sim 10^{-3}$ gm/cm³ $\quad v \sim 500$ km/hr ~ 150 m/sec

$$v^2 \sim 2 \cdot 10^8 \,(\text{cm/sec})^2.$$

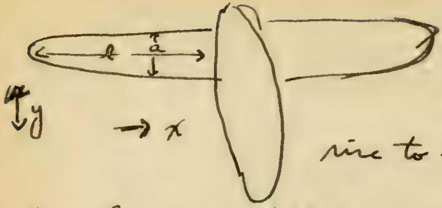
β $\qquad a \sim 5\text{ m} = 5 \cdot 10^2$ cm.

Then lift is $10^{-3} \cdot \pi \cdot 5 \cdot 10^2 \cdot 2 \cdot 10^8$ ($\beta\cos\alpha \ldots$) dynes

$\beta \cdot 10^8$ ( ) dynes/cm.

100 ( ) kg-wt./cm

if $l = 10$ m. $\qquad \sim 100$ tons.



End effects give rise to a velocity in x-direction. gives rise to end drag. Also get other a profile drag (which is smaller).

Still have pot. flow, $\frac{du}{dy} = \frac{dv}{dx}$.

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \frac{\partial u}{\partial y} = \frac{1}{v}\frac{du}{dt}$$

u at trailing edge $\quad u_t = \int_{\text{front}}^{\text{back}} \frac{\partial v}{\partial x}\,dy$

$$u_{t_{\text{upper}}} - u_{t_{\text{lower}}} = \frac{\partial}{\partial x}\int_{\text{upper}} v\,dy - \frac{\partial}{\partial x}\int_{\text{lower}} v\,dy$$

$$= \frac{\partial}{\partial x}\,(\text{circulation}(x))$$

Behind wing, get a

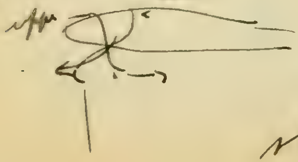
$$= \frac{\partial(\Gamma(x))}{\partial x}.$$

surface of discontinuity where velocities on upper & lower sides are in opposite directions.

$\Gamma = 0$ at end of wing, then increases to some max. value;

$\therefore \frac{\partial \Gamma}{\partial x}$ is positive. $\therefore\ u_{t_{\text{upper}}} \rightarrow \quad u_{t_{\text{lower}}} \leftarrow$.

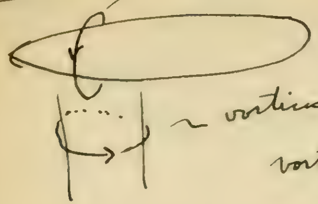
get vortices with axes in direction of motion of plane.



Behind wing have vortex lines in direction of motion (y-direction)

strength $= \frac{d\Gamma}{dx}\,dx$ at x

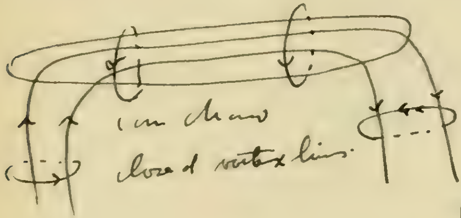
April 29 circulation around wing



~ vortices behind wing

vortex lines in direction of flight.

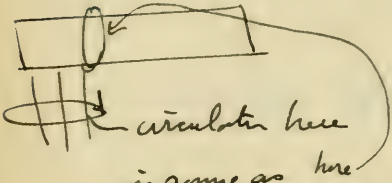
draw lines in direction of $\nabla \times \underline{v}$; $\nabla \cdot \nabla \times \underline{v} = 0$, hence the vortex line never end



strength of vortex line : $d\Gamma = \Gamma' dx$.

velocity produced by one vortex filament $= \dfrac{d\Gamma}{2\pi r}$

$\left[d\Gamma = \oint v\,ds = 2\pi r v \right.$.



velocity produced by vortex fil. at $x = \xi$ is $\dfrac{d\Gamma_\xi}{2\pi(x-\xi)} = \dfrac{\Gamma'(\xi)\,d\xi}{2\pi(x-\xi)}$

$\therefore$ total downward vel. at $x = \displaystyle\int \frac{\Gamma'(\xi)\,d\xi}{2\pi(x-\xi)}$

$= w(x)$.

This would be so if vortex lines went straight from $+\infty$ to $-\infty$; but effectively we have only half of this, since vortex lines go away at wing.

$\therefore$ $w(x) = \dfrac{1}{4\pi} \displaystyle\int \frac{\Gamma'(\xi)\,d\xi}{x-\xi}$. downward velocity.

Now Force is ⊥ velocity of airstream; but we now have a downward component of velocity

∴ force is ↗ instead of ↑; can resolve into two comps. ↳ lift & drag.

$$\text{Ratio } \frac{\text{drag}}{\text{lift}} = \frac{w}{v_\infty}.$$

$$\text{but lift} = \rho v_\infty \Gamma$$

$$\therefore \text{drag} = \rho w \Gamma.$$

$$\text{Total lift} = \rho v_\infty \int \Gamma \, dx$$

$$\text{total drag} = \rho \int w(x) \Gamma(x) \, dx.$$

Want large lift, small drag, i.e. $\int \Gamma \, dx$ a max.,

$$D = \int \Gamma w \, dx \text{ a min.}$$

i.e. make distribution of Γ so that drag (D) is a min.

can do this by changing width of wing, or shape.

Suppose

Given increase γ of Γ over a small distance β must give same change of D wherever applied.

$$\text{Change of } D = \gamma \beta \, w(x_0) + \int \Gamma \, \delta w \, dx$$

if change Γ by γ at x_0

$$\text{but } \delta w(x) = \frac{1}{4\pi}\left[\frac{\gamma}{(x-x_0)} - \frac{\gamma}{x-(x_0+\beta)}\right].$$

$$\int \Gamma(x)\, \delta\omega(x)\, dx = \frac{\gamma}{4\pi} \int_{-\ell}^{\ell} \Gamma(x) \left(\frac{1}{x-x_0} - \frac{1}{x-x_0-\beta} \right) dx$$

$$= \frac{\gamma}{4\pi} \left(\int_{-\ell}^{\ell} \frac{\Gamma(x)}{x-x_0}\, dx - \int_{-\ell-\beta}^{\ell-\beta} \frac{\Gamma(x+\beta)}{x-x_0} \right) dx$$

if $\Gamma(\pm\ell) = 0$:

$$= -\frac{\gamma}{4\pi} \int_{-\ell}^{\ell} \frac{\beta\Gamma'(x)\, dx}{x-x_0} = -\frac{\gamma\beta}{4\pi}\left(-\omega(x_0)\right) 4\pi$$

$$= \gamma\beta\, \omega(x_0).$$

$$\therefore \ \delta D = 2\gamma\beta\,\omega(x_0).$$

if this is to be the same everywhere, we want

$\omega(x_0) = \text{const}$. surface of discontinuity moves down all together with same velocity

Now want $\int \frac{\Gamma'(\xi)\, d\xi}{x-\xi}$ indep^t of x.

this requires an elliptic distrib. of Γ:

$$\Gamma = \sqrt{1 - \left(\frac{\xi}{\ell}\right)^2}$$

[if $\ell = 1$, $\Gamma' = \dfrac{-\xi}{\sqrt{1-\xi^2}}$, $\displaystyle\int_{-1}^{1} \frac{-\xi\, d\xi}{(x-\xi)(\sqrt{1-\xi^2})}$.

$$= \int_0^{\pi} \frac{\cos\varphi \sin\varphi\, d\varphi}{(x-\cos\varphi)\sin\varphi} = \int_0^{\pi} \frac{\cos\varphi\, d\varphi}{x-\cos\varphi}$$

$$= \int_0^{\pi} \left(1 - \frac{x}{x-\cos\varphi}\right) d\varphi$$

$$= \pi$$

$\leftarrow = 0$ if $|x| < 1$.

To achieve this, make a (width of wing) vary elliptically,

can do this : two ellipses.

Betz → increase in drag for rectangular shape 4%.

for 8:1 length to width ratio.

in between shapes, drag only a few %. greater than for elliptical.

if $\Gamma = \Gamma_0 \sqrt{1 - x^2/l^2}$ $\quad w = \frac{\Gamma_0}{4l} = v_\infty \frac{\pi a (\pi\alpha + f\cos\alpha)}{4l}$

$= v_\infty \cdot \frac{\pi}{4} \cdot \left(\frac{a}{l}\right)\left(\pi\alpha + f\cos\alpha\right)$

[$\frac{w}{v_\infty} \sim a/l$ small

$\therefore D \ll L$.

Total lift $\sim v_\infty^2 a l$

Drag $\sim v_\infty^2 a^2$ $\qquad D/L \sim a/l$.

∴ Drag independent of length.

long wing desirable [physically, all because drag arises

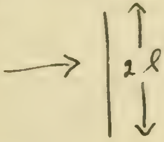
from ends of wing

May 1 1942

We found: "induced" resistance does not depend on length, only on width of wing; but lift depends on length as well (prop. to it). Resistance ~ Γ
Lift ~ Γ. $\therefore$ if increase lift by tilting wing, increase resistance as square of lift.

Discuss now "Profile Drag"

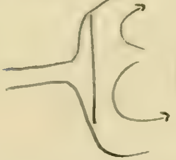
simple case: plate $\perp$ to liquid.



No lift – by symmetry
but high resistance.

Naïve potential theory gives:

but this would give no drag – symmetrical on both sides. But we know this ain't so – but will get separation at ends of plate



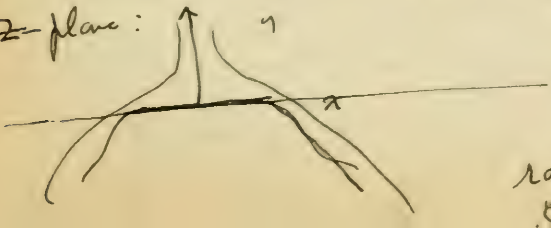
practically at rest here;
some vortices

Introduce w: $\frac{dw}{dz} = u - iv$

$w = \varphi + i\psi$

Let $\zeta = \frac{dz}{dw} = \frac{u+iv}{|v|^2}$.

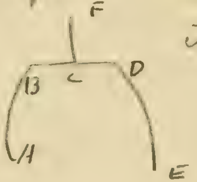
z-plane:



radius vector to a point in the ζ-plane has direction of velocity

in z-plane. magnitude is its reciprocal.

real axis in w-plane [$\psi = 0$] corresponds to stream line:

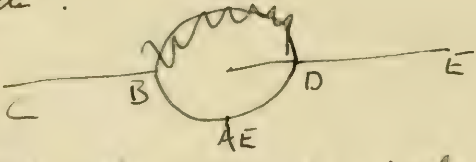


in z-plane. Let C be at origin of w-plane; − real axis corr. to F–C, + real axis to both CBA + CDE.

w-plane:

ζ-plane: we know velocity from C to D is in $+^x$ direction, + goes from 0 to some finite + value; ∴ ζ goes from ∞ to a finite value. Also C–B has vel. in −x direction. along AB + DE we know magnitude of velocity by Bernoulli's theorem. If no vortices inside ABDE, then

can apply Bernoulli theorem; p constant in "dead water" – hence pressure constant along AB + DE; ∴ velocity there same on AB + DE as at B + D. Hence in ζ-plane, get a circle:



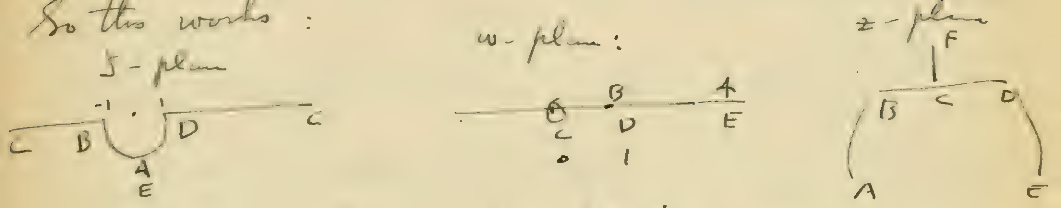
take this as unit circle. Then problem is to map this onto + real axis in w-plane. this is done by: $\frac{2}{\sqrt{w}} = \zeta + \frac{1}{\zeta}$. if ζ is real, & > 1, then $\sqrt{w} < 1$, from 0 to 1

also if $\zeta < -1$, ~~ζ~~ $w < 1$.

also if $\zeta = e^{i\alpha}$, $\frac{2}{\sqrt{w}} = 2\cos\alpha$, $w = \frac{1}{\cos^2\alpha}$,

which goes from 1 to ∞. $(-\pi < \alpha < 0)$.

So this works:



$\frac{dz}{dw} = \zeta$ but $\frac{2}{\sqrt{w}} = \zeta + \frac{1}{\zeta}$;

$z = \int \zeta \, dw$ i.e. $\zeta = \frac{1}{\sqrt{w}} \pm \sqrt{\frac{1}{w} - 1}$

as $\frac{1}{\sqrt{w}} \to \infty$, $\zeta \to \infty$, $\therefore$ take + square root.

Now $\overline{CD} = l = \int_0^1 dw\left(\frac{1}{\sqrt{w}} + \sqrt{\frac{1}{w} - 1}\right)$

but $\int dw\left(\frac{1}{\sqrt{w}} + \sqrt{\frac{1}{w} - 1}\right) = 2\sqrt{w} + \sin^{-1}\sqrt{w} + \sqrt{w(1-w)}$.

$\therefore l = 2 + \sin^{-1} 1 = 2 + \pi/2$

length of plate $= 2l = 4 + \pi$.

if ~~w is imaginary~~, can also write: $z = 2\sqrt{w} + i\ln(\sqrt{w} + \sqrt{w-1}) + \pi/2 - i\sqrt{w(w-1)}$.

for large w, $z \approx 2\sqrt{w} - i\sqrt{w(w-1)}$.

$\therefore$ $x = 2\sqrt{w}$, $y = -w$ } which gives a parabola -

How is velocity along AB, DE related to velocity at infinity?

at F, $w = -\infty$, $\therefore \zeta = -i$ as it should. (i.e. directed downward, & magnitude 1. $\therefore$ velocity at surface of discontinuity $= v_\infty$ (we have really used units of vel. at infinity.

Can calculate resistance: $F = \int (p_{front} - p_{back})\, dx$.

$p_{back} = p_B$; $\quad p_f(x) = -\frac{1}{2}\rho \int [v^2(x) - v^2(B)]$ ← this is unit

$$= -\frac{1}{2}\rho \int_{-(2+\pi/2)}^{+(2+\pi/2)} \left(\frac{1}{\zeta^2} - 1\right) d\frac{z}{i}$$

$$= -\frac{1}{2}\rho \int_{0}^{\infty} \left(\frac{1}{\zeta^2} - 1\right) \zeta\, dw$$

$$= 2\cdot\tfrac{1}{2}\rho \int_0^1 \left(\frac{1}{\zeta} - \zeta\right) dw$$

$$= \rho \int_0^1 2\sqrt{\tfrac{1}{w} - 1}\; dw = \pi\rho$$

$\therefore F = \pi\rho$ but what are the units? actually $F = \rho\cdot v_\infty^2 \cdot \pi \cdot \left(\frac{2\ell}{4+\pi}\right)$ since force is prop. to length. Hence drag coeff is, if

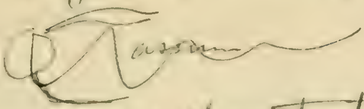
$F = \frac{1}{2}\rho v_\infty^2 \cdot 2\ell \cdot C_D$, $\quad C_D = \frac{2\pi}{4+\pi} \sim .85$.

Observed $C_D \sim 1.3$.

about as good as to be expected, in view of bad approx —

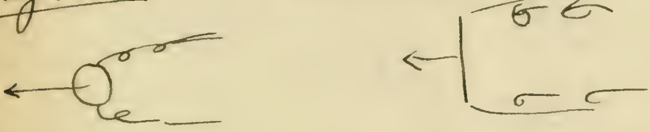
this done by Helmholtz.

Levi-Civita: in general for a body, assume separation at some point



Bethe thinks a sort of self consistent calculation would be good — assume a separation pt., calculate potential flow, then use Karman's theory to find separation pt. —

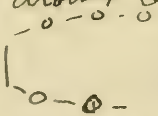
May 4 '42 | Karman Vortex trail.



vortices in neighborhood of surface of discontinuity

A surface of disc. may be considered as a set of vortex lines

Vortex lines are grouped around centres; get discrete vortex centres & vortices.



small vortices would be unstable — only one stable configuration of vortices along two parallel lines (really we have parabolic lines, in case of plate)

Karman studied the case of ∥ lines.

Suppose we have vortices along a line; this is stable if they are equidistant. If one is removed

a little; vortex on right makes it move up, on left, down; but it is closer to one than other, hence it moves out of line. Hence a linear set of vortices is in unstable equilibrium. Actually, always have two sets; one from each side of body.

Suppose two sets:
of opposite direction

lower row produces a vel. to right on upper
upper " " " " right on lower
∴ whole thing moves to right.

No force up or down from one to next. Hence equilibrium

Now displace one of ~~lower~~ upper vortices to left. It will still move away; but not so much; upward velocity due to near upper is partly compensated by downward vel. due to lower near one. Hence still unstable.

But now space ~~them~~ differently:

if put them at ½-way points, can find a value of ratio $\frac{\text{dist betw vortices}}{\text{" " rows}}$ which gives equilibrium.

∴

Handb. der Exp.-Ph. 4, 1, 155

Ratio $\frac{h}{l} = .281$ ← $\left(\sinh \frac{\pi h}{l} = 1\right)$

h = dist. between rows
l = " " vortices.

In actual flow, vortices separate alternately from upper & lower surfaces

Turns out that $\bar{u} = \frac{\Gamma}{2l} \tanh \frac{\pi h}{l} = \frac{\Gamma}{2\sqrt{2}\, l}$.

Drag $D = L\rho\left[\Gamma/l\,(u - 2\bar{u})\right]h + \frac{\Gamma^2}{2\pi l}\Big]$

(L = length, ρ = density, u = vel. of body, $\bar{u}$ = velocity of vortex trail.)

d = width of plate
L = length of plate

$D = C_D \frac{1}{2}\rho u^2 L \cdot d$

$C_D = 2\frac{l}{d}\left(.794\frac{\bar{u}}{u} - .314\left(\frac{\bar{u}}{u}\right)^2\right)$.

agrees within exp. error

Kármán & Rubach made measurements on plate & cylinder.

	plate	cylinder	theory
	$d = 1.75$	$d = 1.5$	
$h =$	3 cm	1.8 cm	
$l =$	4.8 "	6.4 "	
$h/l =$	.305	.28	.281
$\bar{u}/u$:	.20	.14	
drag C_D :	.80	.46 → calculated	

($\frac{\bar{u}}{u}$ is indep. of spee[d], h/l " " ", drag coeff. " ")

→ vortices move slowly in comparison with body —

} not bad.

D'Alembert's Paradox

No net force, in potential flow: $\underline{F} = -\int_{\text{surface}} p \, d\underline{\sigma}$

$$= \frac{1}{2}\rho \int v^2 d\underline{\sigma}$$

$$= \frac{1}{2}\rho \left[\int_{\text{sphere at } \infty} v^2 d\sigma - \int_{\text{volume between surfaces}} \nabla v^2 \, d\tau \right.$$

but $\nabla v^2 = \underline{v} \, \nabla \cdot v - \underline{v} \times \nabla \times \underline{v}$

if incompressible: $\nabla \cdot v = 0$

if irrotational: $\nabla \times v = 0$

$$\therefore \int d\tau \, \nabla v^2 = 0$$

but $\int_{\text{sphere at } \infty} v^2 d\underline{\sigma} = 0$ also, since

$$v = \nabla \varphi$$

$$\varphi = v_\infty \cdot x + \text{const.} \frac{c}{x} + \frac{1}{x^2} \ldots$$

(source.

$$\therefore v = v_\infty \underline{i} + \frac{c}{r^2} \cdot \frac{r}{r}$$

$$v^2 = v_\infty^2 + \frac{c}{r^2} \cdot 2\frac{x}{r} v_\infty$$

but $v_\infty^2 = \text{const.}$, $\int \text{const. } d\sigma = 0$

and $c = 0$ if no sources.

$$\therefore \int v^2 d\underline{\sigma} = 0$$

and $\underline{F} = 0$.

May 6

Compressible fluids:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{v} = 0$$

or: $$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = -\rho \nabla \cdot \underline{v}$$

Eq'n. of motion still the same:

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla) \underline{v} = -\frac{1}{\rho} \nabla p$$ (without viscosity)

Need another equation relating p and $\rho - \rho_0$.

Can get this by consideration of energy:

Change of internal energy per unit mass:

$dE + p\,dV$ = energy received by heat conduction and viscosity.

But heat conduction is rather a slow process; also viscous forces are small & to be considered only at boundaries;

Hence $T dS = dE + p\,dV \simeq 0$.

or: $\frac{dS}{dt} = 0$ entropy conserved for a small element. [Doesn't mean that entropy is constant everywhere – but that each element "carries its entropy with it"]

"adiabatic"

[For very slow motion, heat conduction enough to keep temperature constant "isothermal".]

For gases, for adiabatic behavior:

$$p = \alpha \rho^{\gamma} \qquad \gamma = \frac{c_p}{c_v} = 5/3 \text{ for}$$

monatomic gases

7/5 diatomic

$\gamma \; {>1 \atop <7/5}$ polyatomic

If entropy is constant over whole system for all time (which happens except in shock waves)

Define: $\left(\frac{\partial p}{\partial \rho}\right)_s = c^2$

For Gas: $\frac{dp}{d\rho} = \alpha \gamma \rho^{\gamma - 1}$

$$= \gamma \frac{p}{\rho} = \gamma R T.$$

Define compressibility $K = \frac{1}{?} = \frac{d\rho/\rho}{dp}$

$$\therefore c^2 = \frac{K}{\rho}.$$

Continuity eq'n. $\frac{\partial p}{d t} + \underline{v} \cdot \nabla p = -\rho c^2 \nabla \cdot v.$

---

1. Small disturbance: $\underline{v}$ small, changes in ρ small.

then can neglect 2nd order terms.

$$\frac{\partial \underline{v}}{\partial t} = -\frac{\nabla p}{\rho}$$

$$\frac{\partial p}{\partial t} = -\rho c^2 \nabla \cdot \underline{v}$$

Now can consider $\rho = \rho_0$, since terms on right

are already 1st order.

$$\therefore \frac{\partial^2 p}{\partial t^2} = \frac{\partial}{\partial t}\left(-\rho_0 c_0^2 \nabla\cdot v\right) =$$

$$= -\rho_0 c_0^2 \, \nabla\cdot \frac{\partial v}{\partial t}$$

$$= -\rho_0 c_0^2 \, \nabla\cdot\left(-\frac{\nabla p}{\rho_0}\right)$$

$$= c_0^2 \nabla^2 p$$

wave eq'n.
sound waves.

this is Acoustics.

Simple cases: plane disturbance.

$$p(x,t).$$

$$p = f(x - c_0 t) + g(x + c_0 t).$$

plane waves to right — plane wave to left

If have piston in tube:

get waves to right:

$$p = f(x - c_0 t)$$

$$\frac{\partial p}{\partial x} = -\frac{1}{c_0}\frac{\partial p}{\partial t}$$

But $\dfrac{\partial u}{\partial t} = -\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial x} = -\dfrac{1}{c_0\rho_0}\dfrac{\partial p}{\partial t}$

$$\therefore u = \frac{p}{c_0\rho_0}$$

$c_0 = 330$	m/sec	air
1470	"	water
5000	"	iron

water: $\rho_0 c_0 = 1.5\cdot 10^5$ g/cm²/sec

$$\therefore p = 1.5\cdot 10^5 \cdot u$$

if pressure is measured in bars, $p = .15\, u$.

Air: $\rho_0 c_0 = 40$; $p \sim 40u$ dyn/cm – very small pressures

---

Scale: energy of voice $\sim 10^{-6}$ erg/cm²/sec; but energy $\sim \rho u^2$ $\therefore$ $\rho u^2 c_0 \sim 10^{-6}$

$$u^2 \sim \frac{10^{-6}}{40} \sim 2.5 \cdot 10^{-8}$$

$u \sim 10^{-4}$ cm/sec.

$\therefore$ $p \sim .01$ dyn/cm²..

Spherical sound waves:

$$\nabla^2 p = \frac{1}{c_0^2}\ddot{p}$$

polars:

$$\frac{1}{r}\frac{d^2}{dr^2}(rp) = \frac{1}{c_0^2}\ddot{p}$$

or

$$\frac{\partial^2}{\partial r^2}(rp) = \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}(rp)$$

$$\therefore \; rp = f(r - c_0 t) + g(r + c_0 t).$$

Or if u is radial velocity:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\nabla p = -\frac{1}{\rho_0}\frac{\partial p}{\partial r}$$

$$= -\frac{1}{\rho_0}\left[-\frac{f+g}{r^2} + \frac{f'+g'}{r}\right]$$

$$\therefore \; u = \frac{1}{c_0 \rho_0}p + \frac{1}{\rho_0}\frac{1}{r^2}\int(f+g)\,dt$$

velocity has one term like plane waves, important at large dist; but at small distances have the

"afterflow" term. Origin is because a given radial velocity means larger transport of matter at large distances than small; hence fluid at small distances must try to follow after — For incompressible fluid get only the "afterflow term"

May 8] We had, for small disturbances:

$$\nabla^2 p - \frac{1}{c^2}\ddot{p} = 0$$

$$p \sim \frac{f(r-ct)}{r} \quad \text{for spherical case.}$$

$$\text{radial velocity} \quad u = p/\rho_0 c_0 + \frac{1}{r^2}\int f\,dt$$

$$\text{Energy} = \rho u^2 c$$

if afterflow is negligible $\sim \frac{1}{r^2}$

2nd approximation "hydraulic" approx.

stationary flow thru channel of changing width.

cross-sectional area is slowly-varying f'n of distance

$f(x)$ = area.

flow thru a cross-section/unit time

$$= \rho u f = \text{constant.} \quad \text{continuity}$$

$$u\frac{du}{dx} = -\frac{1}{\rho}\frac{dp}{dx} \quad \text{(eq. of motion)}$$

entropy S = constant (cons. of energy)

[for a gas: $p = \alpha\rho^\gamma$]

From 1st eq'n: $\frac{d\rho}{\rho} + \frac{du}{u} + \frac{df}{f} = 0$ ∴ local sound velocity

$dp = + d\rho\, c^2$ ← by def.

2nd eq'n $= -\rho u\, du$ (c a f'n of x).

$$\therefore \frac{d\rho}{\rho} = -\frac{u\,du}{c^2}.$$

$$\therefore \left(-\frac{u}{c^2} + \frac{1}{u}\right) du + \frac{df}{f} = 0$$

$$\frac{du}{u}\left(1 - u^2/c^2\right) + \frac{df}{f} = 0$$

if $u << c$: $\frac{du}{u} + \frac{df}{f} = 0$ $uf = $ const.

(incompressible approx)

if u is increased, change in f produces bigger change in velocity than before;

if $u > c$, increase in cross section causes increase in velocity of gas!

From $\frac{1}{\rho}\frac{dp}{dx} = -u\,du$

get $\int \frac{dp}{\rho} = -\frac{1}{2}u^2 = w$

For gases $\int \frac{\alpha\gamma\rho^{\gamma-1}\, d\rho}{\rho} = \alpha\gamma \int \rho^{\gamma-2}\, d\rho$

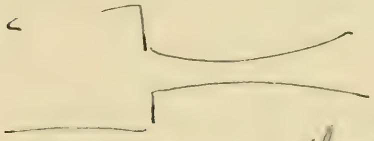
$= \alpha\gamma \frac{\rho^{\gamma-1}}{\gamma-1} = -\frac{1}{2}u^2 = w$

$$\therefore \quad \omega = \frac{\alpha \gamma \rho^{\gamma-1}}{\gamma-1} = \frac{dp/d\rho}{\gamma-1} = \frac{c^2}{\gamma-1}.$$

$$\therefore \quad c^2 - c_1^2 = -\tfrac{1}{2}(\gamma-1)(u^2 - u_1^2)$$

if $u_1 = 0$, $\quad c^2 + \frac{\gamma-1}{2} u^2 = c_1^2$ which is a f'n of temperature only

if $u << c$



velocity increases, then decreases;

For more flow: velocity reaches velocity of sound at smallest crossection, then increases (pressure is very low on right)

if $u = c$: $\quad c_2^2\left(1 + \frac{\gamma-1}{2}\right) = c_1^2$

$$c_2^2 = \frac{2}{\gamma+1} c_1^2$$

but $c^2 \sim \rho^{\gamma-1} \sim p^{\frac{\gamma-1}{\gamma}}$

$$\therefore \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \frac{2}{\gamma+1}$$

$$\therefore \left(\frac{p_2}{p_1}\right) = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}.$$

Another approximation. Riemann theory.

One-dimensional: $u(x)$ p, ρ etc. functions of x & t only.

[More special because of
More general because not
restricted to stationary state]

Continuity: $\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = -\rho \frac{\partial u}{\partial x}$

Eq of motion: $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x}$.

$\frac{dp}{d\rho} = c^2$ [local sound velocity]

Let $\omega = \int \frac{dp}{\rho}$

$$d\rho = \frac{dp}{c^2} = \rho \frac{d\omega}{c^2}$$

continuity: $\therefore \frac{\rho}{c^2}\frac{\partial \omega}{\partial t} + u \frac{\rho}{c^2} \frac{\partial \omega}{\partial x} = -\rho \frac{\partial u}{\partial x}$

or $\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} = -c^2 \frac{\partial u}{\partial x}$

eq. of motion: also $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{\partial \omega}{\partial x}$

Let $d\sigma = \frac{d\omega}{c} = \frac{dp}{c\rho}$

$= \frac{c \, d\rho}{\rho}$

$\therefore \sigma = \int \frac{c}{\rho} d\rho$

($c = c(\rho)$ if entropy is fixed)

or in terms of σ:

$$\begin{cases} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -c \frac{\partial \sigma}{\partial x} \\ \frac{\partial \sigma}{\partial t} + u \frac{\partial \sigma}{\partial x} = -c \frac{\partial u}{\partial x} \end{cases}$$

$$\frac{\partial}{\partial t}(u+\sigma) + u \frac{\partial}{\partial x}(u+\sigma) + c \frac{\partial}{\partial x}(u+\sigma) = 0$$

$$\frac{\partial}{\partial t}(u+\sigma) + (u+c) \frac{\partial}{\partial x}(u+\sigma) = 0$$

Let $u + \sigma = r$

" then $r = f(x - (c+u)t)$.

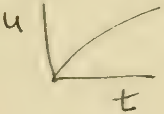
i.e. it is propagated with a velocity $c + u$.

Likewise $(u - \sigma) = s$ is propagated toward $-x$ with the velocity $c - u$.

Suppose we begin moving a piston to right in a pipe with a prescribed motion. If pipe is infinite, all waves go from left to right; i.e. ~~r~~-waves

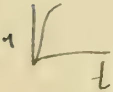
If we push to right, u at piston is $+$, and c corresponds to a compressed gas, so c increases. The later waves have a greater velocity of propagation $c + u$ than the earlier ones — will finally "catch up" with ~~late~~ earlier ones.

So "steepness" of front of wave gets more abrupt.



at piston.

farther on



sufficiently far away.

discontinuous change of material velocity & pressure at front of wave
"shock wave"

May 11

Shock wave: surface of discontinuity of all quantities referring to gas.

discontinuity of u, p, ρ is propagated with velocity U into undisturbed gas –

Apply conservation laws to region enclosing shock front.

Heat conduction is large; viscosity also large (but ordinary viscosity etc. theory doesn't really apply, if wave front is very steep)

Consider shock front at rest, fluid moves ~~L to R~~ R to L.

i.e. velocities relative to shock front).

u_1 = velocity of gas on right = U

$u_2 = U - u$ (u = material velocity on left).

2 | 1 (undisturbed) | Continuity: $\rho_1 u_1 = \rho_2 u_2$ [pressure]

disturbed 1 | momentum conservation: $\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$

energy: E_1 = internal energy / unit mass

$$E_1 + \frac{1}{2} u_1^2 - p_1/\rho_1 = E_2 + \frac{1}{2} u_2^2 - p_2/\rho_2$$

These are independent of material. Eq'n of state is a 4th eq'n:

pu = work done by pressure / cm² / sec.

$\frac{pu}{\rho u}$ = " " per gm.

$$E = E(p, \rho).$$

Suppose we know $\rho_1 p_1 u_1 E_1$

Can get explicit sol'n only for simple eq'ns of state.

In case of ideal gas:

$$E = c_v T$$

$$\left(\frac{c_v}{R}\right)\frac{p}{\rho}$$

$\frac{c_v}{R} = 3/2$ (monatomic)

$5/2$ (diatomic)

Enthalpy: $E + p/\rho = \left(\frac{c_v}{R}+1\right) p/\rho$

$$= \frac{c_p}{R}\cdot p/\rho$$

$$= \beta\, p/\rho$$

$\frac{c_p}{R} = \frac{c_v}{R}+1$

$\beta = \gamma/\gamma-1$

From 2nd eq'n:

$$\rho_1 u_1 (u_1 - u_2) = p_2 - p_1$$

From 3d:

$$\beta\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) = u_1^2 - u_2^2 = (u_1-u_2)(u_1+u_2)$$

$$= \frac{p_2 - p_1}{\rho_1 u_1}(u_1+u_2)$$

& get finally: $\frac{u_2}{u_1} = \frac{(2\beta-1)p_1 + p_2}{(2\beta-1)p_2 + p_1} = \frac{\rho_1}{\rho_2}$

Now $p_2 > p_1$ hence $u_2 < u_1$

(else no shock wave)

if p_2 increases indefinitely,

then $\frac{u_2}{u_1} \rightarrow \frac{1}{2\beta-1} \leftarrow \frac{\rho_1}{\rho_2}$

or $\rho_2/\rho_1 = 2\beta-1$

Hence very high ~~temperature~~

entropy $S \sim p/\rho^\gamma$ increases indefinitely

S ~ log p/ρ^γ.

diatomic gas, $2\beta - 1 = \frac{\gamma+1}{\gamma-1}$

$= 4, 6$

monatomic ↗ diat

$$\frac{u_1-u_2}{u_2} = \frac{(2\beta-2)(p_2-p_1)}{(2\beta-1)p_2+p_1} = \frac{p_2-p_1}{\rho_1 u_1^2}$$

$$\therefore (2\beta-1)p_2 = \rho_1 u_1^2(2\beta-2)+p_1$$

since $p_2 > p_1$,

must have $\rho_1 u_1^2 > \frac{\beta}{\beta-1} p_1$

$= \gamma p_1$

$u_1^2 > \gamma \frac{p_1}{\rho_1} = c_1^2$.

∴ shock wave velocity is always greater than sound velocity.

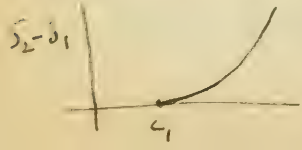
likewise can show that $u_2 < c_2$

velocity of shock wave relative to compressed gas is less than sound.

so sound waves ahead of shock wave are caught by it.
" " behind " " catch it.

if p_2 very large: $p_2 = \frac{2\beta-2}{2\beta-1}\rho_1 U^2$

With a given state of matter in front of shock wave, one parameter determines the whole thing – e.g. by U.



$u_2 = \frac{U}{2\beta-1}$

$u = U - u_2 = U\cdot\frac{2\beta-2}{2\beta-1}$

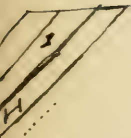
$\therefore p_2 = \frac{2\beta-1}{2\beta-2}\rho_1 u^2$

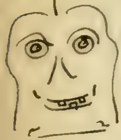
or $p_2 = \rho_1 U u$.

May 13 $\frac{\partial}{\partial t}(\sigma + u) + (c+u)\frac{\partial}{\partial x}(\sigma + u) = 0$

solution is not $\sigma + u = f(x - (c+u)t)$ as was asserted, because $c+u$ is a f'n of x & t.

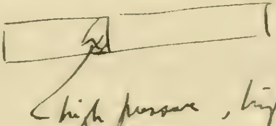
But a value of $\sigma + u$ at $x_0\, t_0$ is propagated in such a way that $x = x_0 + \int_{t_0}^{t} (c+u)_{t'}\, dt'$.





Detonation:

detonator

Chemical reaction takes place rapidly, solid becoming gas + energy: propagates down length of explosive:


high pressure, high temp.

energy E consists of heat + internal energy initially

internal energy → heat energy.

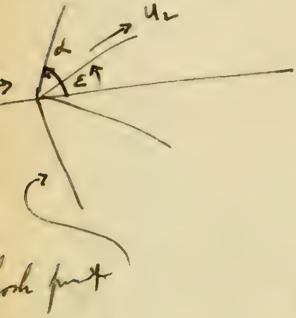
continuity eq'n:

May 15

wedge moving thru air with velocity greater than sound :

$U_1 > c_1$

shock front is formed in front making larger angle



At shock front, velocity has two components, ⊥ & ∥ to front; we have $U_1 \tan = U_2 \tan$

$$\therefore U_1 \cos\alpha = U_2 \cos(\alpha - \varepsilon)$$

normal comp: $u_1 = U_1 \sin\alpha$

$u_2 = U_2 \sin(\alpha - \varepsilon)$

u_1 & u_2 are related by shock front eq'n's:

$$u_2 = \frac{u_1 + (2\beta - 2)\, c_1^2/u_1}{2\beta - 1} . \qquad \beta = c_p/R$$

$$\text{or} \quad u_2/u_1 = \frac{1 + (2\beta-2)\, c_1^2/u_1^2}{2\beta - 1}$$

$$= \frac{\tan(\alpha - \varepsilon)}{\tan\alpha}$$

if ε is small: $= 1 - \dfrac{\varepsilon}{\sin\alpha\cos\alpha}$.

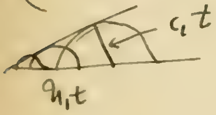
$$= 1 - \frac{2\beta-2}{2\beta-1}\left(1 - \frac{c_1^2}{u_1^2}\right)$$

or $1 - \dfrac{c_1^2}{u_1^2} = \dfrac{2\beta-1}{2\beta-2} \cdot \dfrac{\varepsilon}{\ldots}$.

1st approx: $u_1 = c_1$

and $\sin\alpha \approx \frac{c_1}{U_1}$ "Mach angle"

(shock front is locus of sound waves emitted by wedge)



if $\frac{U_1}{c_1} = 1 + \delta \qquad \delta \ll 1$

$$1 - \frac{c_1^2}{u_1^2} = \frac{2\beta-1}{2\beta-2} \cdot \frac{\varepsilon}{\cos\alpha \sin\alpha} = 1 - \frac{1-2\delta}{\sin^2\alpha}$$

$$= 2\delta - \cos^2\alpha = \frac{\varepsilon'}{\cos\alpha} \qquad \varepsilon' = \frac{(2\beta-1)}{(2\beta-2)}$$

Can't get sol'n for either α too big or α too small

Minimum value of $\delta = \frac{3}{2^{5/3}} \varepsilon'^{2/3}$

---

low velocities: shock wave far from projectile.

Q.M.

$$[P_x\, x] = i\hbar$$

$$[M \times M] = i\hbar \underline{M}$$

$R_0(T)$

$P(T)$

$P = p \cdot \ell \cdot \pi d$

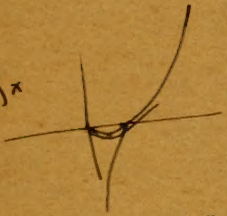
$I^2 R$

$R = R_0 \cdot \frac{\ell}{A} = R_0 \cdot \frac{\ell 4}{\pi d^2} =$

$I^2 \frac{4\ell}{\pi d^2} R_0(T) = \ell \cdot \pi d \cdot P(T)$

$K^2 = \frac{4 I^2}{\pi^2 d^3} = \frac{P(T)}{R_0(T)} = f(T)$

$x^2 \lg x$



$K = \frac{2 I}{\pi d^{3/2}}$

$T = t$

$\lim_{n \to \infty} x^{1/n}$

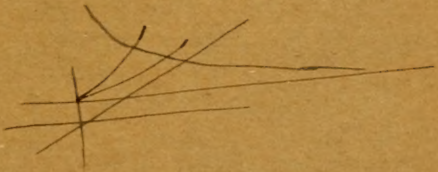
$x^2 \lg x = \frac{x^2}{(\lg x)^{-1}} \to \frac{2x}{-(\lg x)^{-2} \cdot \frac{1}{x}} \to -2x^2 (\lg x)^2$

$x^{1/n}$

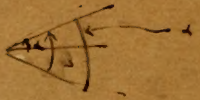
$\frac{\lg x}{x^{-2}} \to \frac{1/x}{-2x^{-3}} \to 0$

$x^{-1} \quad \lg x \quad x \quad x^2 \quad \ldots$

$\lg \quad e^{\frac{1}{x}} \quad x \quad e^{x} \quad e^{x^2}$



$CH_3CH_2CH(NH_2)COOH$



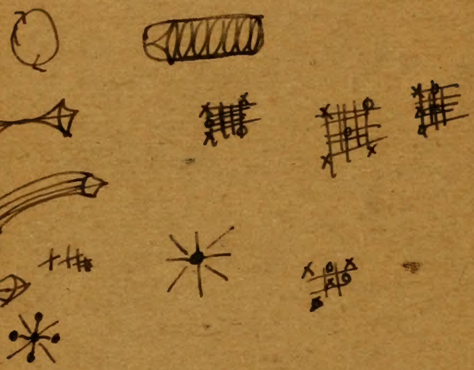
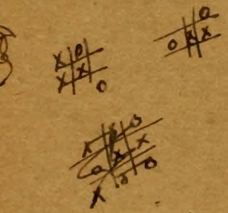
owers

4π

owers

$\pi \cdot \alpha^2 = \beta$

ABCDEFGHIJK L M N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